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FORM MSDC01

MATHEMATICS TEST

No scratch paper version

Last typeset: September 26, 2021

MATHEMATICS TEST

Time — 170 minutes

66 Questions

Directions: Each of the questions or incomplete statements below is followed by five suggested answers or completions. In each case, select the one that is best and then circle the corresponding answer on this sheet.

Computation and scratch work should be done on a separate sheet of paper.

In this test:

1. All logarithms with an unspecified base are natural logarithms, that is, with base e.
2. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of integers, rational numbers, real numbers, and complex numbers, respectively.

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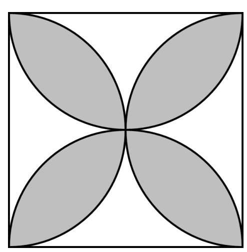
1. $\displaystyle\int_{-2}^{2} \sqrt{16-4x^2}\, dx =$

   (A) 2 (B) 4 (C) 2π (D) 4π (E) 8π



2. Four semicircular arcs are inscribed in a square as shown in the figure above. Find the ratio of the shaded area to the area of the square.

   (A) $\frac{1}{2}(\pi-2)$ (B) $\frac{1}{4}(\pi-2)$ (C) $\frac{1}{4}(\pi-1)$ (D) $\frac{1}{2}(4-\pi)$ (E) $\frac{1}{4}(4-\pi)$

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3. The line $y = x + 1$ is tangent to which of the following curves at $x = 1$?

(A) $y = \sqrt{x}$
(B) $y = \sqrt{x} + 1$
(C) $y = \sqrt{x} - 1$
(D) $y = 2\sqrt{x}$
(E) $y = 2\sqrt{x} - 1$

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4. What are the most specific conditions under which the statement $\big(P \wedge (P \to Q)\big) \to Q$ is true?

(A) If and only if P is true
(B) If and only if Q is true
(C) If and only if P and Q have the same truth value
(D) For all truth values of P and Q
(E) For no truth values of P and Q

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5. Suppose f and g are continuously differentiable functions with the following properties:

$f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$.

$f'(x) > 0$ for all $x \in \mathbb{R}$.

$g'(x) < 0$ for all $x \in \mathbb{R}$.

Which of the following functions is NOT necessarily monotonic?

(A) $\big(g(x)\big)^2$ (B) $f(x) - g(x)$ (C) $f(x)g(x)$ (D) $\dfrac{f(x)}{g(x)}$ (E) $g\big(f(x)\big)$

---

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on the real numbers. Which of the following ensures that $\lim_{x\to-\infty} f(x) = \infty$?

(A) For all $\varepsilon < 0$, there exists a $\delta < 0$ such that $x < \delta$ implies $f(x) < \varepsilon$.
(B) For all $\varepsilon > 0$, there exists a $\delta < 0$ such that $x > \delta$ implies $f(x) > \varepsilon$.
(C) For all $\varepsilon > 0$, there exists a $\delta < 0$ such that $x < \delta$ implies $f(x) > \varepsilon$.
(D) For all $\delta > 0$, there exists an $\varepsilon < 0$ such that $x > \delta$ implies $f(x) > \varepsilon$.
(E) For all $\delta < 0$, there exists an $\varepsilon > 0$ such that $x < \delta$ implies $f(x) > \varepsilon$.

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7. Suppose z is a complex number such that $|z| = 8$. Which of the following is equal to $\dfrac{z}{4}$? (Here $\overline{z}$ stands for the complex conjugate of z.)

(A) $\dfrac{z}{16}$ (B) $\dfrac{\overline{z}}{16}$ (C) $\dfrac{1}{16\overline{z}}$ (D) $\dfrac{16}{z}$ (E) $\dfrac{16}{\overline{z}}$

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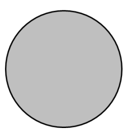
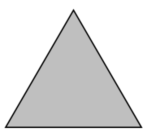
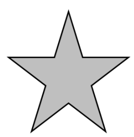
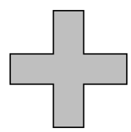
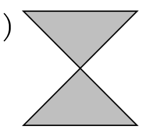
8. For which of the following shapes does the set of rotation and reflection symmetries form an abelian group? (Assume that sides and angles that appear to be congruent are in fact congruent.)

(A)  (B)  (C)  (D)  (E) 

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9. Let $g(x) = \int_3^{x^2} \cos(\sqrt{t})\, dt$. What is the value of $g'(\pi)$?

(A) -2π (B) $-\pi$ (C) -2 (D) -1 (E) 0

---

10. Let $P = (3, -1, 2, 5)$ and $Q = (1, 2, 2, 4)$ be two points in $\mathbb{R}^4$. Which of the following points lies on the line in $\mathbb{R}^4$ connecting P and Q?

(A) $(5, 5, 2, 3)$
(B) $(1, 5, 3, -1)$
(C) $(-1, 3, 2, 0)$
(D) $(-2, 6, 3, 1)$
(E) $(-3, 8, 2, 2)$

---

11. If $f(x) = x^{1/x}$, what is $f'(2)$?

(A) $\frac{\sqrt{2}}{4} + \log 2$ (B) $\frac{\sqrt{2}}{4} \log 2$ (C) $\frac{\sqrt{2}}{4}(1 - \log 2)$ (D) $\frac{\sqrt{2}}{4}(1 + \log 2)$ (E) $\frac{\sqrt{2}}{4}(-1 + \log 2)$

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12. Brian is playing a crane game where the chance of winning a plush toy is $\frac{1}{3}$ each time. What is the probability that it takes him exactly 5 tries to win 3 plush toys?

(A) $\frac{2}{81}$ (B) $\frac{8}{81}$ (C) $\frac{4}{243}$ (D) $\frac{10}{243}$ (E) $\frac{40}{243}$

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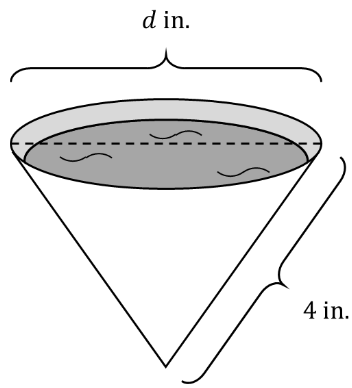
13. A supplier is manufacturing disposable paper cups in the shape of a right circular cone. The slant height of each cone is to be 4 inches long. What should be the diameter of the open circular base of the cup, so that it holds the maximum possible volume of water?



(A) $\frac{4\sqrt{3}}{3}$ (B) $\frac{8\sqrt{3}}{3}$ (C) $\frac{2\sqrt{6}}{3}$ (D) $\frac{4\sqrt{6}}{3}$ (E) $\frac{8\sqrt{6}}{3}$

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14. For how many values of x does $\displaystyle\int_{-3}^{x} t\sqrt{9+t^2}\,dt = 0$?

(A) None (B) One (C) Two (D) Three (E) Four

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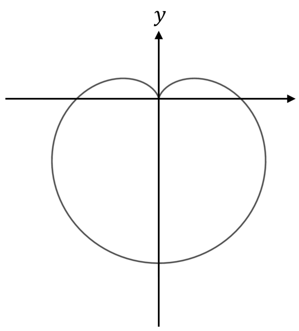
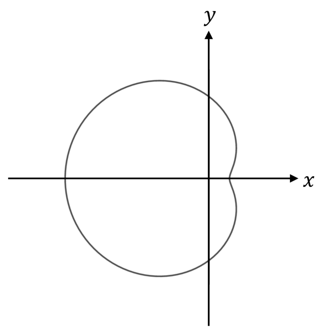
15. Which of the following most closely resembles the graph of the curve defined by the polar equation $r = 1 - 2\cos\theta$?

(A)


(B)


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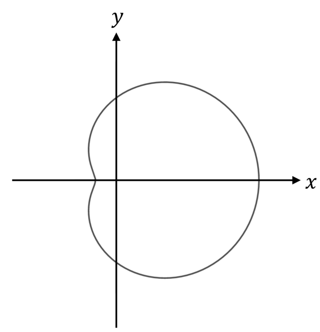

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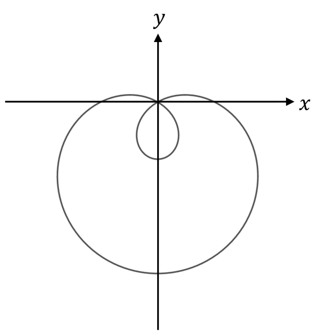
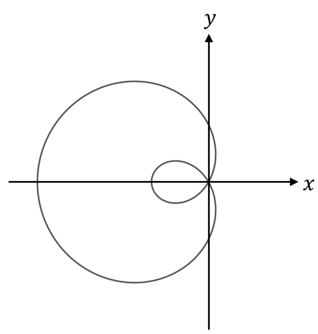


(E)


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16. For a hexadecimal (base 16) number, let the digits $\mathrm{a}, \cdots, \mathrm{f}$ represent the numbers $10, \cdots, 15$ in base 10. Which of the following is a divisor of $14\mathrm{d}_{\text{hex}}$?

(A) $1\mathrm{c}_{\text{hex}}$ (B) 25_{hex} (C) 28_{hex} (D) $2\mathrm{d}_{\text{hex}}$ (E) 33_{hex}

---

17. Let $A = \begin{pmatrix} 1 & -2 & 0 & 2 \\ 3 & 2 & -1 & 4 \\ 1 & 6 & -1 & 0 \end{pmatrix}$. Which of the following is a basis for the null space of A?

(A) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right\}$

(B) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} -4 \\ 2 \\ 8 \\ 4 \end{pmatrix} \right\}$

(D) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

(E) $\left\{ \begin{pmatrix} -4 \\ 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

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18. Which quadrants are contained in the preimage of quadrant III of the complex plane under the mapping $z \mapsto z^3$?

    (A) Quadrant I only
    (B) Quadrant III only
    (C) Quadrants I and III only
    (D) Quadrants I, II, and III only
    (E) Quadrants I, III, and IV only

---

x	0	2	4	6
$f'(x)$	4	1	7	-1
$f''(x)$	-1	3	0	-3

19. The function f is twice differentiable for all real x. Values of $f'(x)$ and $f''(x)$ are given for selected values of x in the table above. Which of the following statements must be true?

    I. f' has a local maximum at $x = 4$.

    II. f has a point of inflection somewhere in the interval $(0, 2)$.

    III. There exists a $c \in [0, 6]$ for which $f''(c) = -4$.

    (A) I only (B) II only (C) III only (D) I and II only (E) II and III only

---

20. Find the volume of the solid formed by revolving about the y-axis the region in the first quadrant bounded by the curves $y = e^{-x^2}$ and $x = 2$.

(A) $\pi(1 - e^{-4})$ (B) $2\pi(1 - e^{-4})$ (C) $\pi(1 + e^{-2})$ (D) $2\pi(1 - e^{-2})$ (E) $2\pi(1 + e^{-2})$

---

21. Suppose a curve C is parametrized by the equations $x = f(t)$ and $y = g(t)$, where f and g are twice-differentiable functions. If $f'(t) \neq 0$, which of the following expressions gives the value of $\dfrac{d^2y}{dx^2}$ when it exists?

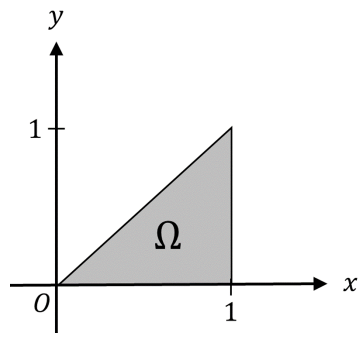
(A) $\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\big(f'(t)\big)^3}$

(B) $\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\big(f'(t)\big)^2}$

(C) $\dfrac{f'(t)g''(t) + g'(t)f''(t)}{\big(f'(t)\big)^3}$

(D) $\dfrac{f'(t)g''(t) + g'(t)f''(t)}{\big(g'(t)\big)^2}$

(E) $\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\big(g'(t)\big)^3}$



22. Evaluate $\iint_\Omega \cos(x^2)\,dx\,dy$, where Ω is the triangular region pictured above.

(A) $\sin 1$ (B) $\cos 1$ (C) $\frac{1}{2}\sin 1$ (D) $\frac{1}{2}(1-\sin 1)$ (E) $\frac{1}{2}(1-\cos 1)$

---

23. If $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, then $f^{-1}(x) =$

(A) $\log\sqrt{\dfrac{x+1}{2}}$ (B) $\log\sqrt{\dfrac{x-1}{2}}$ (C) $\log\sqrt{\dfrac{x+1}{x-1}}$ (D) $\log\sqrt{\dfrac{x-1}{x+1}}$ (E) $\log\sqrt{\dfrac{x+1}{1-x}}$

---

24. Which of these rings has the largest number of units?

(A) $\mathbb{Z}_6$ (B) $\mathbb{Z}_7$ (C) $\mathbb{Z}_8$ (D) $\mathbb{Z}$ (E) $\mathbb{Z} \times \mathbb{Z}$

---

25. Let $y = f(x)$ be a nonzero solution to the differential equation

$$y'' + 4y' + 7y = 0.$$

Which of the following statements must be true?

I. The equation $f(x) = 0$ has infinitely many solutions.

II. $\lim_{x\to\infty} f(x) = 0$.

III. $\lim_{x\to-\infty} \left|f(x)\right| = \infty$.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

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26. During a party, some pairs of people in a room shake hands, while some don't. Nobody shakes the same person's hand more than once. Each person in the room is then asked whether they shook hands with an even or odd number of people. Which of the following statements is true?

(A) The number of people who answered "even" must be even.
(B) The number of people who answered "even" must be odd.
(C) The number of people who answered "odd" must be even.
(D) The number of people who answered "odd" must be odd.
(E) The number of people who answered "odd" could be even or odd.

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27. Let g be the function defined by $g(x, y, z) = z^2 e^{xy}$ for all real x, y, and z. The maximum possible value M of the directional derivative of g at the point $(2, 0, -1)$ in the direction of some vector $\mathbf{u} \in \mathbb{R}^3$ falls within which of the following ranges?

(A) $1 < M < 2$ (B) $2 < M < 3$ (C) $3 < M < 4$ (D) $4 < M < 5$ (E) $M > 5$

---

28. $\dfrac{d^n}{dx^n} \cos x = \sin\left(x + \dfrac{k\pi}{2}\right)$ <u>if and only if</u> which of the following congruences holds?

(A) $k - n \equiv 0 \pmod{2}$
(B) $k - n \equiv 1 \pmod{2}$
(C) $k - n \equiv 1 \pmod{4}$
(D) $k - n \equiv 2 \pmod{4}$
(E) $k - n \equiv 3 \pmod{4}$

---

29. Find the remainder when 6^{293} is divided by 11.

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

---

30. $\lim_{x\to 0} \dfrac{\sin x - \tan x}{x^3} =$

(A) -1 (B) $-\dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$ (E) The limit does not exist.

---

31. Suppose the power series $a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3 + \cdots$ is used to represent the function $f(x) = \dfrac{2x}{x^2+1}$. What is the radius of convergence of this power series?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) ∞

---

32. What is the set of all vectors $\mathbf{v}$ that satisfy the equation $(2,3,4) \times \mathbf{v} = (-3,2,1)$?

(A) $\varnothing$
(B) $\Big\{(-5,-14,13),(8,12,-13)\Big\}$
(C) $\Big\{(5,14,-13),(-8,-6,-12)\Big\}$
(D) $\Big\{(-8,-12,13),(8,6,-12)\Big\}$
(E) $\Big\{(x,y,z) : 3x-2y-z=0\Big\}$

---

33. Suppose x is the smallest positive integer that satisfies the following congruences:

$$\begin{aligned} x &\equiv 1 \pmod{4} \\ x &\equiv 2 \pmod{5} \\ x &\equiv 3 \pmod{7} \end{aligned}$$

Which of the following must be true?

(A) $x \equiv 2 \pmod{9}$
(B) $x \equiv 4 \pmod{9}$
(C) $x \equiv 6 \pmod{9}$
(D) $x \equiv 8 \pmod{9}$
(E) There is no such value of x.

---

34. Consider the following algorithm, which takes an input integer `n>1` and prints a decimal number.

```
input(n)
t := 0
k := 1
s := 1
while (k < n) {
    a := 1/k
    t := t + s*a
    k := k + 2
    s := s * -1
}
output(t)
```

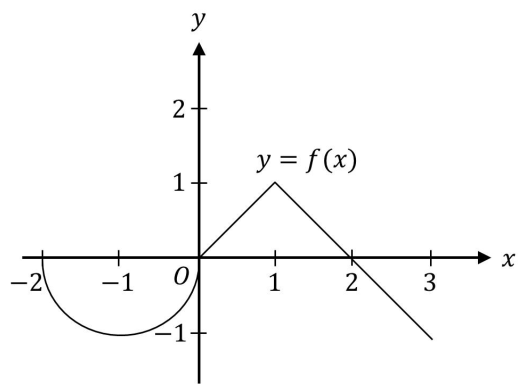
If the input integer is `500`, which of the following will be the output when truncated after the hundredths digit?

(A) `0.59` (B) `0.69` (C) `0.72` (D) `0.78` (E) `0.81`

---

35. Let $y = f(x)$ be a solution to the differential equation $y' = y^3 - 3y + 2$, $y(0) = a$, where $a \in \mathbb{Z}$. For how many values of a is $\lim_{x \to \infty} f(x)$ finite?

(A) One (B) Two (C) Three (D) Four (E) More than four



36. The above graph of $y = f(x)$, defined for $-2 \leq x \leq 3$, consists of a semicircle and two line segments. Define $g(x) = \displaystyle\int_0^x f(t)\,dt$, and let A, B, and C be defined as follows:

$A =$ The maximum value of $g(x)$ for $-2 \leq x \leq 3$

$B =$ The number of inflection points of $g(x)$ for $-2 \leq x \leq 3$

$C =$ The number of points at which $g(x)$ is not differentiable for $-2 \leq x \leq 3$

Which of the following correctly ranks the values of A, B, and C?

(A) $A < B < C$
(B) $A < C = B$
(C) $B < A < C$
(D) $C < A < B$
(E) $C = B < A$

---

37. Suppose $(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Calculate the quantity $\dfrac{1}{\lambda_1\lambda_2} + \dfrac{1}{\lambda_1\lambda_3} + \dfrac{1}{\lambda_2\lambda_3}$.

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$ (C) $\dfrac{2}{3}$ (D) $\dfrac{4}{9}$ (E) $\dfrac{19}{12}$

---

38. If G is a group of order 60, then G does <u>not</u> necessarily have a subgroup of order:

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

---

39. Let $\{a_n\}$ be the sequence defined by $a_n = \left(1 + \dfrac{(-1)^n}{n}\right)^n$. Calculate $\limsup\limits_{n\to\infty} a_n - \liminf\limits_{n\to\infty} a_n$.

(A) 0 (B) $\dfrac{e-1}{e}$ (C) $\dfrac{e-1}{e^2}$ (D) $\dfrac{e^2-1}{e}$ (E) $+\infty$

---

40. Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree at most 3 with real coefficients. Consider the following subspaces of $P_3(\mathbb{R})$:

$$U = \{p \in P_3(\mathbb{R}) : p(0) = 0\}$$

$$V = \{p \in P_3(\mathbb{R}) : p(-1) = p(1) = 0\}$$

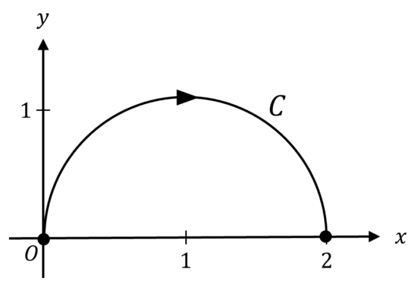
Which of the following statements are true?

I. $U \cap V$ is a subspace of $P_3(\mathbb{R})$.

II. $U \cup V$ is a subspace of $P_3(\mathbb{R})$.

III. $\dim(U) + \dim(V) = \dim(P_3(\mathbb{R}))$.

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III



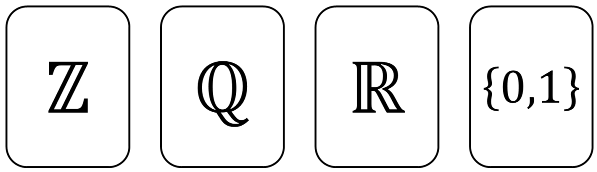
41. Let C be the semicircular path from $(0,0)$ to $(2,0)$ pictured above. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$.

(A) 2 (B) $\frac{8}{3}$ (C) 3 (D) 4 (E) $\frac{16}{3}$

---

42. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$, then $\dfrac{dy}{dx} =$

(A) $\dfrac{1}{2y-1}$ (B) $\dfrac{1}{2y+1}$ (C) $\dfrac{y}{2y-1}$ (D) $\dfrac{2y+1}{y}$ (E) $\dfrac{1-2y}{y}$



43. Neisha has four index cards, each of which has a different set written on it, as shown above. First, Neisha chooses a card at random, and lets A be the set written on that card. Then, she replaces the card and shuffles the cards, chooses a second card at random, and lets B be the set written on the second card. If F is the set of all functions with domain A and codomain B, what is the probability that F is a countable set?

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{3}{8}$ (D) $\dfrac{3}{16}$ (E) $\dfrac{9}{16}$

---

44. Consider the function f defined as

$$f(x) = \begin{cases} \dfrac{c}{x^{5/2}} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1, \end{cases}$$

where c is a real constant. Suppose f is the probability distribution of a continuous random variable X. What is the expected value of X?

(A) 1 (B) 3 (C) 6 (D) 9 (E) ∞

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45. Suppose S is a set of continuous functions on $[3, 5]$ such that for each $f \in S$, the following properties hold:

$f(3) = 2$

$f(5) = 4$

$f'(x) > 0$ for all $x \in [3, 5]$

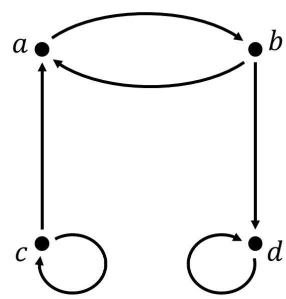
Calculate $\sup_{f \in S} \left\{ \int_2^4 f^{-1}(x)\, dx \right\}$.

(A) 4 (B) 6 (C) 8 (D) 10 (E) 14

---

46. For each integer $n \geq 0$, define $P_n(x) = \displaystyle\int_0^x t^n e^{-t}\,dt$. Which of the following recurrences is satisfied by $P_n(x)$ for all $n \geq 1$?

(A) $P_n(x) = nP_{n-1}(x)$
(B) $P_n(x) = x^n e^{-x} + nP_{n-1}(x)$
(C) $P_n(x) = x^n e^{-x} - nP_{n-1}(x)$
(D) $P_n(x) = -x^n e^{-x} + nP_{n-1}(x)$
(E) $P_n(x) = -x^n e^{-x} - nP_{n-1}(x)$



47. The vertex-edge graph above depicts a relation $\sim$ on the set $S = \{a, b, c, d\}$. For any $x \in S$ and $y \in S$, an arrow drawn from x to y on the graph signifies that $x \sim y$.

What is the minimum number of additional arrows that must be drawn so that the relation represented by the resulting vertex-edge graph is transitive?

(A) Two (B) Three (C) Four (D) Five (E) Seven

---

48. $\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n} =$

(A) 1 (B) e (C) $e^{1/e}$ (D) e^e (E) The limit does not exist.

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49. Michael brought 12 identical cookies to work. In how many ways can he distribute those cookies to his four coworkers so that each coworker gets at least one cookie?

(A) $\binom{11}{3}$ (B) $\binom{12}{3}$ (C) $\binom{12}{4}$ (D) $\binom{13}{4}$ (E) $\binom{15}{3}$

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50. Estimate $\int_0^1 \frac{\sin t}{t}\,dt$ to the nearest thousandth.

(A) 0.942
(B) 0.943
(C) 0.944
(D) 0.945
(E) 0.946

---

51. Which of the following abelian groups of order 360 is NOT isomorphic to the other four?

(A) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
(B) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{15}$
(C) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$
(D) $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_6$
(E) $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30}$

---

52. Let $f : (0,1) \to (0,\infty)$ be a uniformly continuous function. Which of the following statements are true?

I. If $\{x_n\}$ is a Cauchy sequence in $(0,1)$, then $\{f(x_n)\}$ is a Cauchy sequence in $(0,\infty)$.

II. If $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$.

III. If $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in $(0,1)$, then $\{\left|f(x_n) - f(y_n)\right|\}$ is a Cauchy sequence in $(0,\infty)$.

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

---

53. Calculate $\frac{1}{2\pi i}\oint_C \tan z\,dz$, where C is the circle $|z-1|=1$ parametrized counterclockwise.

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

---

54. Suppose A is a 3×3 matrix with the property that $A^3 = A$. Which of the following must be true?

(A) The eigenvalues of A are distinct.
(B) If A is invertible, then $A^T = A$.
(C) A^2 is the identity matrix or the zero matrix.
(D) The trace of A^2 equals the rank of A.
(E) The absolute value of the trace of A^3 equals the rank of A.

---

x	0	1	2	3	4
$f(x)$	1	-5	-7	-2	3

55. Selected values of a polynomial $f(x)$ of degree 4 are given in the table above. What is the value of $f(5)$?

(A) -9 (B) -4 (C) 2 (D) 7 (E) 11

---

56. Let $g(z)$ be an analytic function such that $\left|g(z)\right| = 3$ for all z in the open disk $|z| < 2$. If $g(1) = 3i$, find all possible values of $g(-1)$.

(A) $\{3i\}$
(B) $\{-3i\}$
(C) $\{3i, -3i\}$
(D) $\{3, -3, 3i, -3i\}$
(E) $\{z \in \mathbb{C} : |z| = 3\}$

---

57. Let C_1 be the curve defined by the polar equation $r = \dfrac{\theta}{\theta + 1}$ for $\theta \geq 0$, and let C_2 be the circle defined by the equation $x^2 + y^2 = 1$. Let $C = C_1 \cup C_2$.

Which of the following statements are true with respect to the standard topology on $\mathbb{R}^2$?

I. For any point $P \in C$, there exists an open disk centered at P containing some point $Q \in C$ such that $P \neq Q$.

II. For any collection $\mathcal{F}$ of open disks such that $C \subseteq \bigcup \mathcal{F}$, there exists a finite subcollection $\mathcal{G} \subseteq \mathcal{F}$ such that $C \subseteq \bigcup \mathcal{G}$.

III. For any pair of points $P \in C$ and $Q \in C$, there exists a continuous function $f : [0, 1] \to C$ such that $f(0) = P$ and $f(1) = Q$.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

---

58. $\displaystyle\int_0^\infty \frac{1}{1+e^{ax}}\,dx =$

(A) $\dfrac{1}{a}$ (B) $a\log 2$ (C) $\dfrac{1}{a\log 2}$ (D) $\dfrac{\log 2}{a}$ (E) $\dfrac{a}{\log 2}$

---

59. To the nearest thousand, approximately how many roots does the function $f(x) = e^{-x}\sin(x^2)$ have on the interval $[0, 100]$?

(A) 1000 (B) 2000 (C) 3000 (D) 4000 (E) 5000

---

60. Circle C is tangent to the graph of $y = x^2$ at the origin and has the same curvature as the parabola at the point of tangency. What is the radius of circle C?

(A) $\dfrac{1}{3}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{2}{3}$ (D) $\dfrac{3}{2}$ (E) 2

---

61. Let V be the vector space of continuous functions $\mathbb{C} \to \mathbb{C}$ under pointwise addition and scalar multiplication by complex numbers, and let A be the set of fourth roots of unity in the complex plane. For each $\alpha \in A$, define the set $S_\alpha = \{\cos \alpha z, \sin \alpha z, e^{\alpha z}\}$.

What is the dimension of the subspace of V spanned by $\displaystyle\bigcup_{\alpha \in A} S_\alpha$?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 12

---

62. Which of the following must be true of a function $f : \mathbb{R}^2 \to \mathbb{R}$?

I. If $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist and are continuous at $(0, 0)$, then f is differentiable there.

II. If f has directional derivatives in all directions at $(0, 0)$, then f is differentiable there.

III. If $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ exist at $(0, 0)$, then they are equal there.

(A) None (B) II only (C) I and II only (D) I and III only (E) II and III only

---

63. Consider the matrix equation $QR\mathbf{x} = \mathbf{b}$, where $Q = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ is an orthogonal matrix,

$R = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{pmatrix}$ is an upper triangular matrix, and $\mathbf{b} = \begin{pmatrix} 11 \\ -10 \\ -65 \end{pmatrix}$.

If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, what is the value of x_3?

(A) -6 (B) -3 (C) 3 (D) 9 (E) 12

---

64. Let A and B be ideals of a ring R, and define the ideals $A + B$ and AB as follows:

$$A + B = \{a + b : a \in A, b \in B\}$$
$$AB = \{a_1 b_1 + \cdots + a_n b_n : a_i \in A, b_i \in B, i \in \{1, \ldots, n\}, n \in \{1, 2, \ldots\}\}$$

Which of the following correctly orders $A + B$, AB, and $A \cap B$ via inclusion?

(A) $AB \subseteq A + B \subseteq A \cap B$
(B) $A \cap B \subseteq AB \subseteq A + B$
(C) $A \cap B \subseteq A + B \subseteq AB$
(D) $AB \subseteq A \cap B \subseteq A + B$
(E) $A + B \subseteq A \cap B \subseteq AB$

---

65. For each pair of integers a and b, let the set $\{a + bn : n \in \mathbb{Z}\}$ be denoted $a + b\mathbb{Z}$. Consider the topology $\mathcal{T}$ on $\mathbb{Z}$ whose basis is $\{a + b\mathbb{Z} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Which of the following statements is false?

(A) $(\mathbb{Z}, \mathcal{T})$ is Hausdorff.
(B) $(\mathbb{Z}, \mathcal{T})$ is totally disconnected.
(C) The set $\{0\}$ is closed under $\mathcal{T}$.
(D) No nonempty open sets of $\mathcal{T}$ are finite.
(E) Exactly two sets of $\mathcal{T}$ are both open and closed.

---

66. Let an abelian group M form a left module over a ring R. We say that a subset S of M is a spanning set of M if for every $m \in M$, there exist $\{r_1, r_2, \ldots, r_n\} \subseteq R$ and $\{s_1, s_2, \ldots, s_n\} \subseteq S$ such that

$$m = \sum_{i=1}^{n} r_i s_i.$$

We call this spanning set S minimal if no proper subset of S is a spanning set for S. In addition, S is called a basis for M if $m = 0$ implies $r_1 = r_2 = \cdots = r_n = 0$.

Which of the following statements must be true?

I. If R is finite, then any basis S of M is finite.

II. If S and S' are two distinct minimal spanning sets of M, then $|S| = |S'|$.

III. If R is a field and S is a finite minimal spanning set of M, then S is a basis of M.

(A) II only (B) III only (C) I and II only (D) I and III only (E) II and III only

---

S T O P

If you finish before time is called, you may check your work on this test.

Answers

1. D	11. C	21. A	31. B	41. B	51. D	61. B
2. A	12. B	22. C	32. A	42. A	52. E	62. A
3. D	13. E	23. C	33. D	43. D	53. B	63. A
4. D	14. C	24. B	34. D	44. B	54. D	64. D
5. C	15. E	25. C	35. D	45. D	55. A	65. E
6. C	16. B	26. C	36. D	46. D	56. A	66. B
7. E	17. D	27. B	37. C	47. D	57. C	
8. E	18. E	28. C	38. E	48. B	58. D	
9. A	19. E	29. B	39. D	49. A	59. C	
10. E	20. A	30. B	40. A	50. E	60. B	

Score Conversion Chart*

Raw Score	Scaled Score	Percentile		Raw Score	Scaled Score	Percentile
970	66	99		660	32	50
960	65	99		650	31	48
950	64	99		640	30	46
940	63	99		630	29	44
930	62	99		620	28	41
920	61	98		610	27	39
910	60	96		600	26	37
900	59	94		590	25	34
890	58	92		580	24	31
880	57	90		570	22 – 23	29
870	55 – 56	89		560	21	27
860	54	87		550	20	25
850	53	86		540	19	22
840	52	84		530	18	20
830	51	83		520	17	18
820	50	81		510	16	16
810	49	80		500	15	14
800	48	78		490	14	12
790	47	77		480	13	11
780	46	75		470	11 – 12	9
770	44 – 45	74		460	10	8
760	43	71		450	9	6
750	42	70		440	8	5
740	41	67		430	7	4
730	40	66		420	6	3
720	39	63		410	5	2
710	38	61		400	4	2
700	37	59		390	3	1
690	36	57		380	2	1
680	35	55		370	1	0
670	33 – 34	53		360	0	0

* The above Score Conversion Chart is based entirely on my own observations and estimations. It should *NOT* be taken as any guarantee of a particular score on the actual GRE®Math Subject Test.