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FORM MSDC01

MATHEMATICS TEST
SOLUTIONS

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This test is NOT an official GRE® Mathematics Subject Test.

1. $\int_{-2}^{2} \sqrt{16-4x^2}\,dx =$

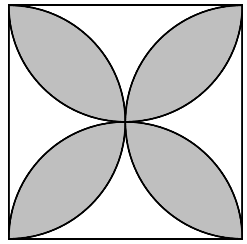
(A) 2 (B) 4 (C) 2π (D) 4π (E) 8π

Solution:

We could work this out with trig substitution (make the substitution $x = 2\sin u$), but the easiest thing to do is to think geometrically:

$$\begin{aligned} y &= \sqrt{16-4x^2} \\ y^2 &= 16-4x^2 \\ 4x^2+y^2 &= 16 \\ \frac{x^2}{4}+\frac{y^2}{16} &= 1 \end{aligned}$$

So the integrand is the upper half of an ellipse with semimajor axis $a = 4$ and semiminor axis $b = 2$. The area of this half-ellipse is $\frac{1}{2}\pi ab = 4\pi$.

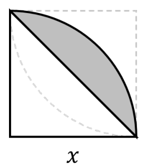


2. Four semicircular arcs are inscribed in a square as shown in the figure above. Find the ratio of the shaded area to the area of the square.

   (A) $\frac{1}{2}(\pi - 2)$ (B) $\frac{1}{4}(\pi - 2)$ (C) $\frac{1}{4}(\pi - 1)$ (D) $\frac{1}{2}(4 - \pi)$ (E) $\frac{1}{4}(4 - \pi)$

   Solution:

   Suppose the square has side length $2x$, so that its area is $4x^2$. Now we can look at half of one of the "petals" as in the following figure.

   The shaded area is the difference of a quarter-circle with area $\frac{1}{4}\pi x^2$ and a right triangle with area $\frac{1}{2}x^2$, giving us $(\frac{1}{4}\pi - \frac{1}{2})x^2$. There are eight of these figures in the overall picture, so the shaded area is $(2\pi - 4)x^2$. Thus the ratio of the shaded area to the area of the square comes out to $\dfrac{(2\pi - 4)x^2}{4x^2} = \dfrac{1}{2}(\pi - 2)$.

3. The line $y = x + 1$ is tangent to which of the following curves at $x = 1$?

(A) $y = \sqrt{x}$
(B) $y = \sqrt{x} + 1$
(C) $y = \sqrt{x} - 1$
(D) $y = 2\sqrt{x}$
(E) $y = 2\sqrt{x} - 1$

Solution:

When $x = 1$, we have $y = 1 + 1 = 2$, so the curve must pass through $(1, 2)$; eliminate A, C, and E. Then, the derivative of $y = \sqrt{x} + 1$ is $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$, which evaluated at $x = 1$ gives us $\frac{1}{2}$. On the other hand, the derivative of $y = 2\sqrt{x}$ is $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x}}$, which evaluated at $x = 1$ gives us 1. Since the line $y = x + 1$ has a slope of 1, the answer must be D.

4. What are the most specific conditions under which the statement $\big(P \wedge (P \to Q)\big) \to Q$ is true?

   (A) If and only if P is true
   (B) If and only if Q is true
   (C) If and only if P and Q have the same truth value
   (D) For all truth values of P and Q
   (E) For no truth values of P and Q

   Solution:

   We can check this with a truth table:

P	Q	$P \to Q$	$P \wedge (P \to Q)$	$\big(P \wedge (P \to Q)\big) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

   Thus the statement is true regardless of the truth values of P and Q. If you think about what the statement *means*, this makes sense: if P is true, and P implies Q, then Q must be true as well.

5. Suppose f and g are continuously differentiable functions with the following properties:

   $f(x) > 0$ and $g(x) > 0$ for all $x \in \mathbb{R}$.

   $f'(x) > 0$ for all $x \in \mathbb{R}$.

   $g'(x) < 0$ for all $x \in \mathbb{R}$.

   Which of the following functions is NOT necessarily monotonic?

   (A) $\big(g(x)\big)^2$ (B) $f(x) - g(x)$ (C) $f(x)g(x)$ (D) $\dfrac{f(x)}{g(x)}$ (E) $g\big(f(x)\big)$

   Solution:

   Let's work out each in turn by differentiation, looking at the signs we get:

   A: $\dfrac{d}{dx}\big(g(x)\big)^2 = 2g(x)g'(x) = 2(+)(-) < 0$

   B: $\dfrac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x) = (+) - (-) > 0$

   C: $\dfrac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x) = (+)(+) + (+)(-)$

   D: $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = \dfrac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2} = \dfrac{(+)(+) - (+)(-)}{(+)^2} > 0$

   E: $\dfrac{d}{dx}g\big(f(x)\big) = g'\big(f(x)\big)f'(x) = g'(+) \cdot (+) = (-)(+) < 0$

   Notice that with option C, $f'(x)g(x)$ is positive while $f(x)g'(x)$ is negative; however, we have no idea which is greater magnitude, so we cannot conclude the sign for any given x. It's entirely possible for this result to be positive for some x and negative for others, so $f(x)g(x)$ does not have to be monotonic.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined on the real numbers. Which of the following ensures that $\lim_{x \to -\infty} f(x) = \infty$?

   (A) For all $\varepsilon < 0$, there exists a $\delta < 0$ such that $x < \delta$ implies $f(x) < \varepsilon$.
   (B) For all $\varepsilon > 0$, there exists a $\delta < 0$ such that $x > \delta$ implies $f(x) > \varepsilon$.
   (C) For all $\varepsilon > 0$, there exists a $\delta < 0$ such that $x < \delta$ implies $f(x) > \varepsilon$.
   (D) For all $\delta > 0$, there exists an $\varepsilon < 0$ such that $x > \delta$ implies $f(x) > \varepsilon$.
   (E) For all $\delta < 0$, there exists an $\varepsilon > 0$ such that $x < \delta$ implies $f(x) > \varepsilon$.

   Solution:

   This one is really about just remembering how limits are defined when involving infinity. Since x needs to approach $-\infty$, it needs to be able to be less than any negative real number δ, since $f(x)$ needs to approach ∞, it needs to be able to be greater than any positive real number ε. Past that, just remember every analysis student's favorite phrase: "For every ε there exists a δ..."

7. Suppose z is a complex number such that $|z| = 8$. Which of the following is equal to $\frac{z}{4}$? (Here $\overline{z}$ stands for the complex conjugate of z.)

(A) $\frac{z}{16}$ (B) $\frac{\overline{z}}{16}$ (C) $\frac{1}{16\overline{z}}$ (D) $\frac{16}{z}$ (E) $\frac{16}{\overline{z}}$

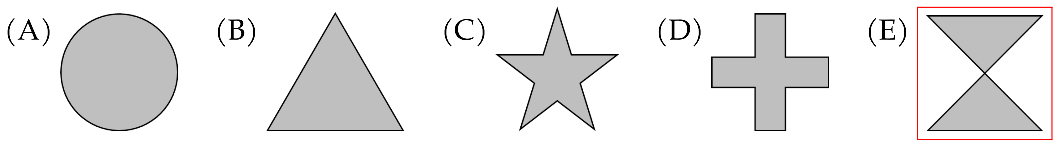
Solution:

Recall that $|z| = \sqrt{z \cdot \overline{z}}$, so we can calculate hence:

$$\begin{aligned} \sqrt{z \cdot \overline{z}} &= 8 \\ z \cdot \overline{z} &= 64 \\ z &= \frac{64}{\overline{z}} \\ \frac{z}{4} &= \frac{1}{4} \cdot \frac{64}{\overline{z}} = \frac{16}{\overline{z}} \end{aligned}$$

If you didn't remember the above equation for $|z|$, though, you could always try letting $z = 8i$ and eliminating the other choices! (You wouldn't want to try $z = 8$ because then you can't tell the difference between z and $\overline{z}$.)

8. For which of the following shapes does the set of rotation and reflection symmetries form an abelian group? (Assume that sides and angles that appear to be congruent are in fact congruent.)



Solution:

Let D_n stand for the dihedral group of order $2n$, which consists of n reflection symmetries and n rotation symmetries $360/n$ degrees apart, including the identity transformation.

For $n = 3$ or greater, D_n is NOT abelian, so this eliminates B (D_3), C (D_5), and D (D_4).

However, E only has two lines of reflection symmetry (one horizontal and one vertical) and two rotations $(0^\circ$ and $180^\circ)$, so its group is D_2. This is isomorphic to the Klein 4-group V_4, which is abelian.

The circle, which has infinitely many reflection and rotation symmetries, has as its symmetry group the orthogonal group $O(n, \mathbb{R})$, but you don't really need to know that — all you need to know is that reflections and rotations (other than 0° or 180°) don't commute in general.

9. Let $g(x) = \int_3^{x^2} \cos(\sqrt{t})\,dt$. What is the value of $g'(\pi)$?

(A) -2π (B) $-\pi$ (C) -2 (D) -1 (E) 0

Solution:

We can use Leibniz's Rule to evaluate this:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f\big(b(x)\big)b'(x) - f\big(a(x)\big)a'(x)$$

Here we have $a(x) = 3$, $b(x) = x^2$, and $f(t) = \cos(\sqrt{t})$. Thus we compute:

$$\begin{aligned} g'(x) &= \cos\left(\sqrt{x^2}\right)\cdot 2x - \cos(\sqrt{3})\cdot 0 \\ &= 2x\cos x \end{aligned}$$

Substituting $x = \pi$, we have $g'(\pi) = 2\pi \cdot -1 = -2\pi$.

10. Let $P = (3, -1, 2, 5)$ and $Q = (1, 2, 2, 4)$ be two points in $\mathbb{R}^4$. Which of the following points lies on the line in $\mathbb{R}^4$ connecting P and Q?

(A) $(5, 5, 2, 3)$
(B) $(1, 5, 3, -1)$
(C) $(-1, 3, 2, 0)$
(D) $(-2, 6, 3, 1)$
(E) $(-3, 8, 2, 2)$

Solution:

The line connecting P and Q can be parametrized as

$$\mathbf{r}(t) = P + (Q - P)t = (3 - 2t, -1 + 3t, 2, 5 - t).$$

This instantly eliminates B and D since their third coordinates are not 2. From here, we look at each remaining option in turn:

- If $3 - 2t = 5$, then $t = -1$, but this would imply that $-1 + 3(-1) = 5$. Eliminate A.
- If $3 - 2t = -1$, then $t = 2$, but this would imply that $-1 + 3(2) = 3$. Eliminate C.
- If $3 - 2t = -3$, then $t = 3$, which also implies that $-1 + 3(3) = 8$ and $5 - 3 = 2$.

11. If $f(x) = x^{1/x}$, what is $f'(2)$?

(A) $\frac{\sqrt{2}}{4} + \log 2$ (B) $\frac{\sqrt{2}}{4} \log 2$ (C) $\frac{\sqrt{2}}{4}(1 - \log 2)$ (D) $\frac{\sqrt{2}}{4}(1 + \log 2)$ (E) $\frac{\sqrt{2}}{4}(-1 + \log 2)$

Solution:

The standard way to work this is to use logarithmic differentiation:

$$\begin{aligned} y &= x^{1/x} \\ \log y &= \log x^{1/x} \\ &= \frac{1}{x} \log x \\ \frac{d}{dx} \log y &= \frac{d}{dx} \left(\frac{1}{x} \log x \right) \\ \frac{y'}{y} &= \frac{1}{x} \cdot \frac{1}{x} + \frac{-1}{x^2} \log x \\ y' &= y \cdot \frac{1}{x^2}(1 - \log x) \end{aligned}$$

Substituting $x = 2$, we have $x = 2^{1/2} \cdot \frac{1}{2^2}(1 - \log 2) = \frac{\sqrt{2}}{4}(1 - \log 2)$.

There's a neat trick that can make this quicker though. It turns out that if f and g are both functions of x, then while f^g is neither a power function nor an exponential function, we can treat it first like a power function and then like an exponential function, and add the two results together:

$$\frac{d}{dx} f^g = g f^{g-1} \cdot f' + f^g \ln f \cdot g'$$

12. Brian is playing a crane game where the chance of winning a plush toy is $\frac{1}{3}$ each time. What is the probability that it takes him exactly 5 tries to win 3 plush toys?

(A) $\frac{2}{81}$ (B) $\frac{8}{81}$ (C) $\frac{4}{243}$ (D) $\frac{10}{243}$ (E) $\frac{40}{243}$

Solution:

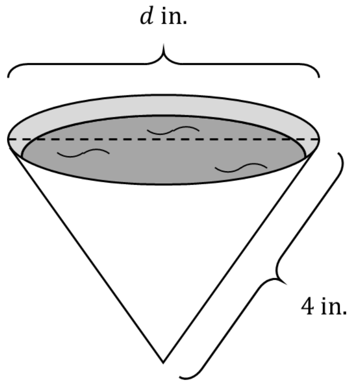
The probability of losing two games and winning three, in that order, is $\left(\frac{2}{3}\right)^2 \cdot \left(\frac{1}{3}\right)^3 = \frac{4}{3^5}$.

However, there are multiple orders in which these games could have happened; the fifth game is always a win, but the first four games consist of two wins and two losses. Thus we need the number of arrangements of the "word" WWLL, which is

$$\frac{4!}{2!2!} = \frac{24}{2 \cdot 2} = 6.$$

Multiplying the above probability by this number of arrangements, we have $6 \cdot \frac{4}{3^5} = \frac{8}{81}$.

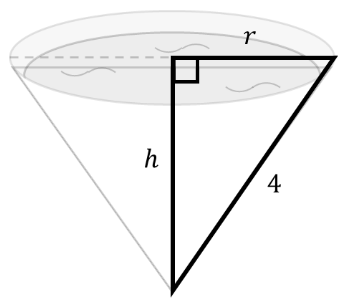
13. A supplier is manufacturing disposable paper cups in the shape of a right circular cone. The slant height of each cone is to be 4 inches long. What should be the diameter of the open circular base of the cup, so that it holds the maximum possible volume of water?



(A) $\frac{4\sqrt{3}}{3}$ (B) $\frac{8\sqrt{3}}{3}$ (C) $\frac{2\sqrt{6}}{3}$ (D) $\frac{4\sqrt{6}}{3}$ (E) $\frac{8\sqrt{6}}{3}$

Solution:

First let's sketch the situation: The volume of the cone is $V = \frac{1}{3}\pi r^2 h$. Since the slant height is 4, we know that $r^2 + h^2 = 4^2$.



At this point, we could either eliminate r or h; however, eliminating r makes the algebra much nicer, so that we don't have to deal with square roots! Let $r^2 = 16 - h^2$, which gives us $V = \frac{1}{3}\pi(16 - h^2)h = \frac{16}{3}\pi h - \frac{1}{3}\pi h^3$.

Differentiating, we have $\frac{dV}{dh} = \frac{16}{3}\pi - \pi h^2$; solving for h, we have $h = \sqrt{\frac{16}{3}}$. This means that $r = \sqrt{16 - \frac{16}{3}} = \frac{32}{3} = \frac{4\sqrt{6}}{3}$. Therefore the *diameter* that maximizes the volume of the paper cup is $d = \frac{8\sqrt{6}}{3}$ inches.

14. For how many values of x does $\int_{-3}^{x} t\sqrt{9+t^2}\,dt = 0$?

(A) None (B) One (C) Two (D) Three (E) Four

Solution:

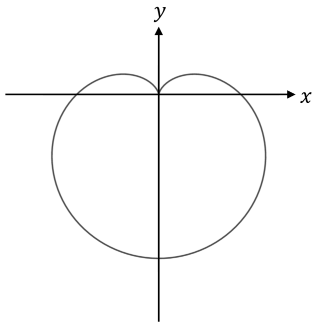
First of all, if we let $x = -3$, then our integral covers no area and we get 0.

Next, suppose $x = 3$. Since t is an odd function and $\sqrt{9+t^2}$ is an even function, their product is odd. This means we're integrating an odd function over a symmetric interval, so we again get 0.

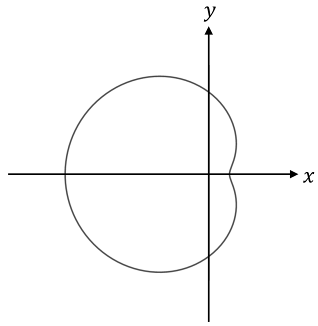
To see that these are the only solutions, notice that by the Fundamental Theorem of Calculus, $\frac{d}{dx}\int_{-3}^{x} t\sqrt{9+t^2} = x\sqrt{9+x^2}$, which only is zero at $t = 0$. This means that the function defined by our above integral only changes direction once, so the two zeros we found are the only ones.

15. Which of the following most closely resembles the graph of the curve defined by the polar equation $r = 1 - 2\cos\theta$?

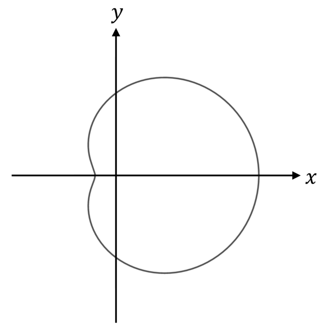
(A)



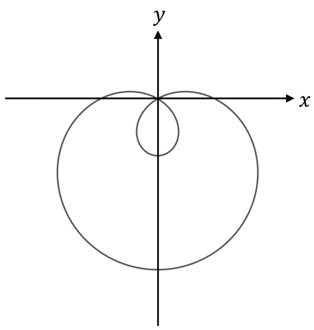
(B)



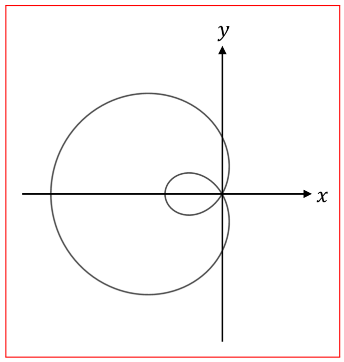
(C)



(D)



(E)



Solution:

First of all, notice that since r is a function of $\cos\theta$, we have

$$r(-\theta) = 1 - 2\cos(-\theta) = 1 - 2\cos\theta = r(\theta).$$

Therefore the curve should be symmetric about $\theta = 0$, which is the horizontal axis. Eliminate A and D.

Now notice that the curve should touch the origin whenever $r = 0$. Setting $1 - 2\cos\theta = 0$, we get $\cos\theta = \frac{1}{2}$, which occurs at $\theta = \frac{\pi}{3}$ and $\theta = -\frac{\pi}{3}$. The only remaining curve that intersects the origin twice is E.

16. For a hexadecimal (base 16) number, let the digits $\mathrm{a}, \cdots, \mathrm{f}$ represent the numbers $10, \cdots, 15$ in base 10. Which of the following is a divisor of $14\mathrm{d}_{\text{hex}}$?

(A) $1\mathrm{c}_{\text{hex}}$ (B) 25_{hex} (C) 28_{hex} (D) $2\mathrm{d}_{\text{hex}}$ (E) 33_{hex}

Solution:

As stated in the directions, the hexadecimal number system has the digits 0-9, as well as the extra digits a, b, c, d, e, and f, which correspond to 10, 11, 12, 13, 14, and 15 in base 10 respectively.

First let's convert $14\mathrm{d}$ to base 10:

$$14\mathrm{d}_{\text{hex}} = 1 \cdot 16^2 + 4 \cdot 16^1 + 13 \cdot 16^0 = 256 + 64 + 13 = 333$$

Now let's convert each of the answer choices to base 10:

A: $1\mathrm{c}_{\text{hex}} = 1 \cdot 16^1 + 12 \cdot 16^0 = 28$

B: $25_{\text{hex}} = 2 \cdot 16^1 + 5 \cdot 16^0 = 37$

C: $28_{\text{hex}} = 2 \cdot 16^1 + 8 \cdot 16^0 = 40$

D: $2\mathrm{d}_{\text{hex}} = 2 \cdot 16^1 + 13 \cdot 16^0 = 45$

E: $33_{\text{hex}} = 3 \cdot 16^1 + 3 \cdot 16^0 = 51$

Since $333 = 3^2 \cdot 37$, the correct divisor is 25_{hex}.

A smart way to go about this would be to factor 333 first and then, seeing that 37 is a factor, convert 37 into base 16 and see that it's 25_{hex}.

17. Let $A = \begin{pmatrix} 1 & -2 & 0 & 2 \\ 3 & 2 & -1 & 4 \\ 1 & 6 & -1 & 0 \end{pmatrix}$. Which of the following is a basis for the null space of A?

(A) $\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \right\}$

(B) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(C) $\left\{ \begin{pmatrix} -4 \\ 2 \\ 8 \\ 4 \end{pmatrix} \right\}$

(D) $\left\{ \begin{pmatrix} 2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

(E) $\left\{ \begin{pmatrix} -4 \\ 2 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 8 \\ 0 \end{pmatrix}, \begin{pmatrix} -6 \\ 1 \\ 0 \\ 4 \end{pmatrix} \right\}$

Solution:

The null space of a matrix A is the set of all vectors $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$. Normally to compute this, we'd augment by a column of zeros and row-reduce, but since this is multiple-choice, there's no reason to do that!

A: This answer is bogus — a 3×4 matrix can't be multiplied by a 3×1 column vector. (In fact, this is the basis of the column space.)

B: Multiplying by the zero vector trivially gives us $\mathbf{0}$, so no surprises here. This could be the answer if the matrix has a full rank of 3, so let's keep going.

C: Multiplying A by this vector does in fact give us $\mathbf{0}$, so we can eliminate B.

D: Multiplying A by either of these vectors also gives us $\mathbf{0}$, and neither is a multiple of the other. Therefore our null space has dimension at least 2; eliminate C.

E: Multiplying A by each of these three vectors gives us $\mathbf{0}$; however, notice that the first vector is the sum of the other two. This means the set is not linearly independent, and therefore can't be a basis. Eliminate E, leaving D as the correct answer.

18. Which quadrants are contained in the preimage of quadrant III of the complex plane under the mapping $z \mapsto z^3$?

(A) Quadrant I only
(B) Quadrant III only
(C) Quadrants I and III only
(D) Quadrants I, II, and III only
(E) Quadrants I, III, and IV only

Solution:

Let a complex number z be represented in polar form as $z = re^{i\theta}$; then by de Moivre's theorem, we have $z^3 = r^3 e^{3i\theta}$. In other words, each complex number with argument θ will be sent to a new complex number with argument 3θ.

Now let's look at where each quadrant goes in turn:

- Quadrant I contains complex numbers with argument $0 < \theta < \frac{\pi}{2}$; these will be sent to complex numbers with argument $0 < 3\theta < \frac{3\pi}{2}$, encompassing quadrants I, II, and III.
- Quadrant II contains complex numbers with argument $\frac{\pi}{2} < \theta < \pi$; these will be sent to complex numbers with argument $\frac{3\pi}{2} < 3\theta < 3\pi$, encompassing quadrants IV, I, and II.
- Quadrant III contains complex numbers with argument $\pi < \theta < \frac{3\pi}{2}$; these will be sent to complex numbers with argument $3\pi < 3\theta < \frac{9\pi}{2}$, encompassing quadrants III, IV, and I.
- Quadrant IV contains complex numbers with argument $\frac{3\pi}{2} < \theta < 2\pi$; these will be sent to complex numbers with argument $\frac{9\pi}{2} < 3\theta < 6\pi$, encompassing quadrants II, III, and IV.

Thus the preimage of quadrant III under the mapping $z \to z^3$ includes quadrants I, III, and IV.

x	0	2	4	6
$f'(x)$	4	1	7	-1
$f''(x)$	-1	3	0	-3

19. The function f is twice differentiable for all real x. Values of $f'(x)$ and $f''(x)$ are given for selected values of x in the table above. Which of the following statements must be true?

I. f' has a local maximum at $x = 4$.

II. f has a point of inflection somewhere in the interval $(0, 2)$.

III. There exists a $c \in [0, 6]$ for which $f''(c) = -4$.

(A) I only (B) II only (C) III only (D) I and II only (E) II and III only

Solution:

I. f' has a local maximum when f'' changes sign from positive to negative.

It's tempting to pick this one, since for $x = 2 \to 4 \to 6$, $f''(x)$ goes positive $\to$ zero $\to$ negative. However, it's possible that the graph of f'' merely touches the x-axis at $x = 4$, goes back up, and then crosses through it some time afterward, say at $x = 5$. All we know is that there has to be a sign change *somewhere* between 4 and 6.

II. f has a point of inflection when f'' changes sign.

At first we might think to use the Intermediate Value Theorem to say that there has to be a sign change between -1 and 3. Note that we did not say that f is twice *continuously* differentiable. Nevertheless, Darboux's theorem says that the derivative of a function still satisfies the intermediate value property, so our sign change is still guaranteed.

III. Again the Intermediate Value Theorem won't really help us here, since the table only shows values of f'' between -3 and 3. However, notice that on the interval $[4, 6]$, the average rate of change of $f'(x)$ is $\dfrac{-1-7}{6-4} = -4$. Since f'' is twice differentiable, f' is differentiable (as well as continuous), so by the Mean Value Theorem, there must exist a $c \in [4, 6]$ such that $f''(c) = -4$.

20. Find the volume of the solid formed by revolving about the y-axis the region in the first quadrant bounded by the curves $y = e^{-x^2}$ and $x = 2$.

(A) $\pi(1 - e^{-4})$ (B) $2\pi(1 - e^{-4})$ (C) $\pi(1 + e^{-2})$ (D) $2\pi(1 - e^{-2})$ (E) $2\pi(1 + e^{-2})$

Solution:

We can use the Shell Method to find the volume:

$$V = \int_0^2 2\pi x f(x)\,dx = \int_0^2 2\pi x e^{-x^2}\,dx$$

Substitute $u = -x^2$ and $du = -2x\,dx$:

$$\begin{aligned} V &= \int_0^{-4} \pi e^u(-du) \\ &= \pi \int_{-4}^{0} e^u\,du \\ &= \pi e^u \Big|_{-4}^{0} \\ &= \pi(1 - e^{-4}) \end{aligned}$$

21. Suppose a curve C is parametrized by the equations $x = f(t)$ and $y = g(t)$, where f and g are twice-differentiable functions. If $f'(t) \neq 0$, which of the following expressions gives the value of $\dfrac{d^2y}{dx^2}$ when it exists?

(A) $\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\big(f'(t)\big)^3}$

(B) $\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\big(f'(t)\big)^2}$

(C) $\dfrac{f'(t)g''(t) + g'(t)f''(t)}{\big(f'(t)\big)^3}$

(D) $\dfrac{f'(t)g''(t) + g'(t)f''(t)}{\big(g'(t)\big)^2}$

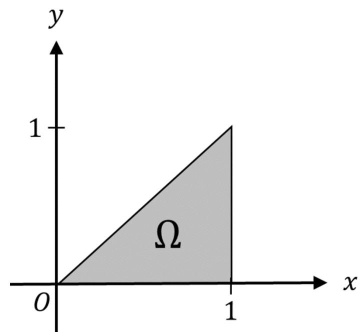
(E) $\dfrac{f'(t)g''(t) - g'(t)f''(t)}{\big(g'(t)\big)^3}$

Solution:

First, the first derivative is $\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$.

The easiest way to remember the second derivative for parametric equations is $\dfrac{d^2y}{dx^2} = \dfrac{dy'}{dx}$:

$$\frac{dy'}{dx} = \frac{\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)}{f'(t)} = \frac{\frac{f'(t)g''(t) - g'(t)f''(t)}{\big(f'(t)\big)^2}}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{\big(f'(t)\big)^3}$$



22. Evaluate $\iint_\Omega \cos(x^2)\,dx\,dy$, where Ω is the triangular region pictured above.

(A) $\sin 1$ (B) $\cos 1$ (C) $\frac{1}{2}\sin 1$ (D) $\frac{1}{2}(1-\sin 1)$ (E) $\frac{1}{2}(1-\cos 1)$

Solution:

In order to compute this integral, we need to switch the order of integration:

$$\begin{aligned}\iint_\Omega \cos(x^2)\,dx\,dy &= \int_0^1\int_0^x \cos(x^2)\,dy\,dx\\ &= \int_0^1 y\cos(x^2)\Big|_0^x\,dx\\ &= \int_0^1 x\cos(x^2)\,dx\\ &= \frac{1}{2}\sin(x^2)\Big|_0^1\\ &= \frac{1}{2}\sin 1\end{aligned}$$

23. If $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$, then $f^{-1}(x) =$

(A) $\log\sqrt{\dfrac{x+1}{2}}$ (B) $\log\sqrt{\dfrac{x-1}{2}}$ (C) $\log\sqrt{\dfrac{x+1}{x-1}}$ (D) $\log\sqrt{\dfrac{x-1}{x+1}}$ (E) $\log\sqrt{\dfrac{x+1}{1-x}}$

Solution:

We find $f^{-1}(x)$ by writing $y = f(x)$, swapping x and y, and then solving for y again.

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{e^x - e^{-x}} \\ x &= \frac{e^y + e^{-y}}{e^y - e^{-y}} \\ x(e^y - e^{-y}) &= e^y + e^{-y} \\ xe^y - xe^{-y} &= e^y + e^{-y} \end{aligned}$$

At this point, we can multiply both sides of the equation by e^y, which allows us to solve for y:

$$\begin{aligned} xe^{2y} - x &= e^{2y} + 1 \\ xe^{2y} - e^{2y} &= x + 1 \\ (x-1)e^{2y} &= x + 1 \\ e^{2y} &= \frac{x+1}{x-1} \\ 2y &= \log\frac{x+1}{x-1} \\ y &= \frac{1}{2}\log\frac{x+1}{x-1} \\ &= \log\sqrt{\frac{x+1}{x-1}} \end{aligned}$$

(*This problem was adapted from GCTM's 1999 Written Test.*)

24. Which of these rings has the largest number of units?

(A) $\mathbb{Z}_6$ (B) $\mathbb{Z}_7$ (C) $\mathbb{Z}_8$ (D) $\mathbb{Z}$ (E) $\mathbb{Z} \times \mathbb{Z}$

Solution:

A unit is an element of a ring that has a multiplicative inverse.

For Z_n, the units are those values of n that are coprime to n.

- For $\mathbb{Z}_6$, the units are 1 and 5.
- For $\mathbb{Z}_7$, the units are $1, 2, 3, 4, 5$, and 6. Note that this means $\mathbb{Z}_7$ is a field (which is always true of $\mathbb{Z}_p$ for a prime p).
- For $\mathbb{Z}_8$, the units are 1, 3, 5, and 7.

For $\mathbb{Z}$, the only units are 1 and -1 — all the other multiplicative inverses are fractions. Then, for $\mathbb{Z} \times \mathbb{Z}$, the multiplicative identity is $(1, 1)$, so the only units are $(1, 1)$, $(1, -1)$, $(-1, 1)$, and $(-1, -1)$.

25. Let $y = f(x)$ be a nonzero solution to the differential equation

$$y'' + 4y' + 7y = 0.$$

Which of the following statements must be true?

I. The equation $f(x) = 0$ has infinitely many solutions.

II. $\lim_{x \to \infty} f(x) = 0$.

III. $\lim_{x \to -\infty} |f(x)| = \infty$.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

Solution:

The characteristic equation of this differential equation is $\lambda^2 + 4\lambda + 7 = 0$, which we can solve with the quadratic formula:

$$\lambda = \frac{-4 \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)} = \frac{-4 \pm \sqrt{-12}}{2} = -2 \pm i\sqrt{3}$$

Since our solutions are nonreal, the general solution to our system of equations is

$$y = c_1 e^{-2t} \cos(\sqrt{3}t) + c_2 e^{-2t} \sin(\sqrt{3}t).$$

I. Since our solution oscillates around 0, there are indeed infinitely many solutions.

II. The e^{-2t} factor means that this is a decaying oscillator, so it indeed tends toward zero. We could prove this if we wanted using the Squeeze Theorem.

III. While the magnitude of $f(x)$ does grow greater as $x \to -\infty$, it also returns back to 0 infinitely often, which means that this limit does not exist.

26. During a party, some pairs of people in a room shake hands, while some don't. Nobody shakes the same person's hand more than once. Each person in the room is then asked whether they shook hands with an even or odd number of people. Which of the following statements is true?

(A) The number of people who answered "even" must be even.
(B) The number of people who answered "even" must be odd.
(C) The number of people who answered "odd" must be even.
(D) The number of people who answered "odd" must be odd.
(E) The number of people who answered "odd" could be even or odd.

Solution:

Each handshake that occurs is between exactly two people, so if we were to add up each person's individual handshake count to make a total, we would count each handshake twice. That is, if Alice and Bob shake hands, then the Alice-Bob handshake will be counted once when Alice gives her count and once when Bob gives his count.

This means the sum of all the handshake counts is double the number of handshakes — in particular, that sum is an even number. However, if an odd number of people answered "odd" to the question, then the sum of all the handshake counts would be odd, so the number of people who answered "odd" must be even.

You might be able to better visualize this if you think of it as a graph theory problem — each person is a vertex, and each handshake is an edge. Then this property says that the number of vertices of odd degree must be even. (This is even sometimes called the "Handshake Theorem" in graph theory.)

27. Let g be the function defined by $g(x, y, z) = z^2 e^{xy}$ for all real x, y, and z. The maximum possible value M of the directional derivative of g at the point $(2, 0, -1)$ in the direction of some vector $\mathbf{u} \in \mathbb{R}^3$ falls within which of the following ranges?

(A) $1 < M < 2$ (B) $2 < M < 3$ (C) $3 < M < 4$ (D) $4 < M < 5$ (E) $M > 5$

Solution:

Recall that the directional derivative in the direction of a unit vector $\mathbf{u}$ is given by $\nabla g \cdot \mathbf{u}$. The maximum possible value is then the gradient ∇g:

$$\nabla g(x, y, z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right) = \left(yz^2 e^{xy}, xz^2 e^{xy}, 2ze^{xy}\right)$$

Thus $\nabla g(2, 0, -1) = (0, 2, -2)$, the magnitude of which is $2\sqrt{2} \approx 2.828$.

28. $\dfrac{d^n}{dx^n}\cos x = \sin\left(x + \dfrac{k\pi}{2}\right)$ if and only if which of the following congruences holds?

(A) $k - n \equiv 0 \pmod 2$
(B) $k - n \equiv 1 \pmod 2$
(C) $k - n \equiv 1 \pmod 4$
(D) $k - n \equiv 2 \pmod 4$
(E) $k - n \equiv 3 \pmod 4$

Solution:

One way to solve this would be to use the sum of angles formula:

$$\sin\left(x + \frac{k\pi}{2}\right) = \sin x \cos\frac{k\pi}{2} + \cos x \sin\frac{k\pi}{2}$$

From here, if $n = 1$, then we have $\dfrac{d}{dx}\cos x = -\sin x = \sin x\cos\dfrac{k\pi}{2} + \cos x \sin\dfrac{k\pi}{2}$. This means we need $\cos\dfrac{k\pi}{2} = -1$ and $\sin\dfrac{k\pi}{2} = 0$, which is true if $k = 2$. Thus $k - n = 1 \equiv 1 \pmod 4$.

Another possibility would be to let $n = 3$, giving us $\dfrac{d^3}{dx^3}\cos x = \sin x = \sin\left(x + \dfrac{k\pi}{2}\right)$. Since $\sin x$ has period 2π, k can be any multiple of 4, and the result follows.

29. Find the remainder when 6^{293} is divided by 11.

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Solution:

By Fermat's Little Theorem, we know that $6^{10} \equiv 1 \pmod{11}$.

This means that $6^{293} = 6^{10\cdot 29+3} = (6^{10})^{29} \cdot 6^3 \equiv 6^3 \pmod{11}$.

Hence could either find the remainder manually when dividing $6^3 = 216$ by 11, or we could aggressively reduce mod 11 whenever possible:

$$6^2 = 36 \equiv 3 \pmod{11}$$
$$6^3 = 6 \cdot 6^2 \equiv 6 \cdot 3 = 18 \equiv 7 \pmod{11}$$

30. $\lim\limits_{x\to 0} \dfrac{\sin x - \tan x}{x^3} =$

(A) -1 (B) $-\dfrac{1}{2}$ (C) 0 (D) $\dfrac{1}{2}$ (E) The limit does not exist.

Solution:

Seeing the indeterminate form $\frac{0}{0}$, we could use L'Hôpital's Rule, but if we do so in its current state, the results will be hairy. Instead, noticing $\sin x$ in the numerator and at least one x in the denominator, let's play with the expression algebraically:

$$\frac{\sin x - \tan x}{x^3} = \frac{\sin x - \frac{\sin x}{\cos x}}{x^3} = \frac{\sin x(1 - \sec x)}{x^3} = \frac{\sin x}{x} \cdot \frac{1 - \sec x}{x^2}$$

This means that we can split our limit into $\lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \dfrac{1-\sec x}{x^2}$. The former limit is 1, which you can check with L'Hôpital's Rule if you wish, so we just need to evaluate the latter. Since $\sec 0 = 1$, let's go ahead and use L'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1 - \sec x}{x^2} \overset{\text{L'H}}{=} \lim_{x\to 0} \frac{\sec x \tan x}{2x}$$

Again we have the indeterminate form $\frac{0}{0}$, but we can split this into $\lim\limits_{x\to 0} \dfrac{\tan x}{x} \cdot \lim\limits_{x\to 0} \dfrac{\sec x}{2}$. The former limit comes out to 1 again by a quick application of L'Hôpital's Rule, and the latter comes out to $\frac{1}{2}$. Thus the overall limit is $\frac{1}{2}$.

31. Suppose the power series $a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3 + \cdots$ is used to represent the function $f(x) = \dfrac{2x}{x^2+1}$. What is the radius of convergence of this power series?

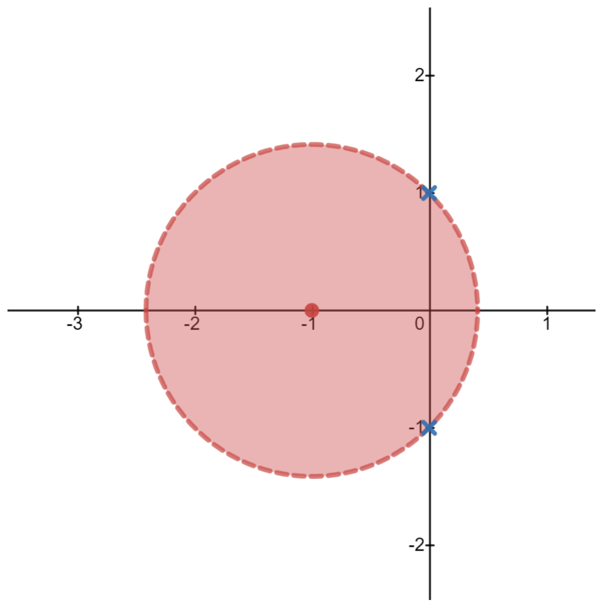
(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) ∞

Solution:

We could go through the work of finding a Taylor series expansion about $x = -1$ and then determining the radius of convergence using the ratio test. However, the easiest way to think about this is to imagine f as a function of a *complex* number:

$$f(z) = \frac{2z}{z^2+1}$$

Note that this function has singularities (simple poles, in fact) at $z = i$ and $z = -i$. Therefore, the *disk* of convergence centered at $z = -1$ can only extend until it hits either of these singularities (which it actually hits both of at the same time).



Thus the radius of convergence is the distance from $z = -1$ to $z = i$, which is $\sqrt{2}$.

32. What is the set of all vectors $\mathbf{v}$ that satisfy the equation $(2, 3, 4) \times \mathbf{v} = (-3, 2, 1)$?

(A) $\varnothing$

(B) $\{(-5, -14, 13), (8, 12, -13)\}$

(C) $\{(5, 14, -13), (-8, -6, -12)\}$

(D) $\{(-8, -12, 13), (8, 6, -12)\}$

(E) $\{(x, y, z) : 3x - 2y - z = 0\}$

Solution:

Remember that for two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ will be perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. However, notice that $(2, 3, 4) \cdot (-3, 2, 1) = (2)(-3) + (3)(2) + (4)(1) = 4$. Since this dot product isn't zero, there's no way that $(2, 3, 4) \times \mathbf{v}$ could possibly be $(-3, 2, 1)$ no matter what $\mathbf{v}$ is.

33. Suppose x is the smallest positive integer that satisfies the following congruences:

$$\begin{aligned} x &\equiv 1 \pmod 4 \\ x &\equiv 2 \pmod 5 \\ x &\equiv 3 \pmod 7 \end{aligned}$$

Which of the following must be true?

(A) $x \equiv 2 \pmod 9$
(B) $x \equiv 4 \pmod 9$
(C) $x \equiv 6 \pmod 9$
(D) $x \equiv 8 \pmod 9$
(E) There is no such value of x.

Solution:

The Chinese Remainder Theorem says that since the moduli are coprime, we can definitely find a solution:

$$x = 1(5 \cdot 7)(5 \cdot 7)_4^{-1} + 2(4 \cdot 7)(4 \cdot 7)_5^{-1} + 3(4 \cdot 5)(4 \cdot 5)_7^{-1}$$

Here, the notation $(5 \cdot 7)_4^{-1}$ stands for the multiplicative inverse of $5 \cdot 7 \pmod 4$.

- $5 \cdot 7 = 35 \equiv -1 \pmod 4$, so its inverse is $-1 \pmod 4$.
- $4 \cdot 7 = 28 \equiv 3 \pmod 5$, so its inverse is $2 \pmod 5$.
- $4 \cdot 5 = 20 \equiv -1 \pmod 7$, so its inverse is $-1 \pmod 7$.

Therefore the solution above gives $x = 1(35)(-1) + 2(28)(2) + 3(20)(-1) = -35 + 112 - 60 = 17$. This solution is unique mod $4 \cdot 5 \cdot 7 = 140$, so it is the smallest positive solution, which means $x \equiv 8 \pmod 9$.

A faster way to do this would be to notice that if $x \equiv 1 \pmod 4$, then x must be odd, and if also $x \equiv 2 \pmod 5$, then x must end in a 7. From here, we could let $x = 10n+7$ and substitute this into the last congruence to find a suitable value of x, or we could just start looking at 3 more than multiples of 7 and keep going until we find one that ends in a 7 while satisfying the first congruence.

34. Consider the following algorithm, which takes an input integer `n>1` and prints a decimal number.

```
input(n)
t := 0
k := 1
s := 1
while (k < n) {
    a := 1/k
    t := t + s*a
    k := k + 2
    s := s * -1
}
output(t)
```

If the input integer is `500`, which of the following will be the output when truncated after the hundredths digit?

(A) `0.59` (B) `0.69` (C) `0.72` (D) `0.78` (E) `0.81`

Solution:

The effect of this algorithm is to approximate the infinite series:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

In particular:

- `k` is the index of the term being computed, which increases by `2` each time.
- `n` is the maximum index added to the summation.
- `s` is the sign of the term, which is repeatedly flipped by multiplying by `-1`.
- `a` is the absolute value of the term, which is then multiplied by `s` to produce the alternating series.
- `t` is the total of all the terms so far.

Since the value of `n` given is `500`, the last term appended is `-1/499`. Then the error for the series is bounded above by the absolute value of the next term, which would have been `1/501`, guaranteeing that our approximation is correct to the nearest hundredth.

The value of this series can be found by remembering the Taylor series for $\arctan x$:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

The value of this series evaluated at $x = 1$ is $\frac{\pi}{4} \approx$ `0.78`.

35. Let $y = f(x)$ be a solution to the differential equation $y' = y^3 - 3y + 2$, $y(0) = a$, where $a \in \mathbb{Z}$. For how many values of a is $\lim\limits_{x\to\infty} f(x)$ finite?

(A) One (B) Two (C) Three (D) Four (E) More than four

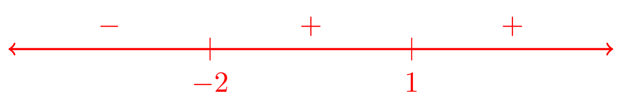
Solution:

What we're looking for is the equilibrium solutions to the differential equation, which are found when $y' = 0$. By inspection, $y' = 0$ when $y = 1$; we can divide $y^3 - 3y + 2$ by $y - 1$ to factor it, for example with synthetic division.

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

Therefore we have $y' = (y-1)(y^2 + y - 2) = (y-1)(y-1)(y+2)$, so $y = 1$ and $y = 2$ are the equilibrium solutions.

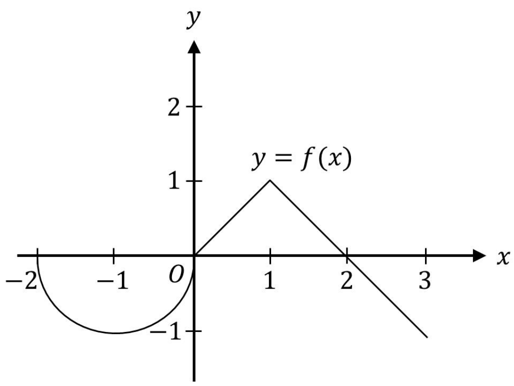
We can then build a sign chart for y':



Now we can classify what happens for initial conditions in these regions.

- If $y(0) < -2$, then $y' < 0$, and the solution will tend toward $-\infty$.
- If $-2 < y(0) < 1$, then $y' > 0$, and the solution will tend toward a limit of 2.
- If $y(0) > 1$, then $y' > 0$, but the solution will tend toward $+\infty$.

Thus $\lim\limits_{x\to\infty} f(x)$ is finite for the initial condition $y(0) = a$ if $a = -1, 0, 1$, or 2.



36. The above graph of $y = f(x)$, defined for $-2 \le x \le 3$, consists of a semicircle and two line segments. Define $g(x) = \displaystyle\int_0^x f(t)\,dt$, and let A, B, and C be defined as follows:

$A =$ The maximum value of $g(x)$ for $-2 \le x \le 3$

$B =$ The number of inflection points of $g(x)$ for $-2 \le x \le 3$

$C =$ The number of points at which $g(x)$ is not differentiable for $-2 \le x \le 3$

Which of the following correctly ranks the values of A, B, and C?

(A) $A < B < C$
(B) $A < C = B$
(C) $B < A < C$
(D) $C < A < B$
(E) $C = B < A$

Solution:

Let's look at each of A, B, and C:

- For A, note that if we integrate *backwards* from 0, we get a *positive* result. In particular, $g(-2) = \displaystyle\int_0^{-2} g(x)\,dx = \frac{\pi}{2} \approx 1.57.$
- For B, g has an inflection point wherever g'' changes sign. Since $g'' = f'$, we see from the graph that f' changes sign at $x = -1$ (where f switches from decreasing to increasing) and $x = 1$ (where f switches from increasing to decreasing), so g has two inflection points.
- For C, g is not differentiable wherever g' is not defined. Since $g' = f$, we see from the graph that f is defined for all x in the interval $-2 \le x \le 3$. (Don't let the sharp corners fool you — that's where f, not g, fails to be differentiable!) So, $C = 0$.

Ordering these, we have $C < A < B$.

37. Suppose $(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Calculate the quantity $\frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}$.

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{9}$ (E) $\frac{19}{12}$

Solution:

The quantity we want can be rewritten as $\frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1\lambda_2\lambda_3}$.

The sum of the eigenvalues is the trace of the matrix, which is $3+2+3=8$. We could also then find the determinant of the matrix, which is the product of the eigenvalues, but we can actually figure out the eigenvalues directly by looking at the matrix:

- The rows have a constant sum of 4, so 4 is one of the eigenvalues (corresponding to the vector $(1,1,1)^T$).
- By inspection, subtracting $3I$ from the main diagonal will give two identical rows of $\begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, so 3 is one of the eigenvalues.
- Since the trace is 8, the remaining eigenvalue must be 1.

Therefore the product of the eigenvalues is $3 \cdot 4 \cdot 1 = 12$, so or desired quantity is $\frac{8}{12} = \frac{2}{3}$.

38. If G is a group of order 60, then G does <u>not</u> necessarily have a subgroup of order:

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Solution:

Since G has order 60, Cauchy's theorem for groups says that any prime p that divides 60, G has an element of order p. Since $60 = 2^2 \cdot 3 \cdot 5$, G must have elements of order 2, 3, and 5, and therefore these elements will therefore generate cyclic subgroups of their respective orders.

Then, Sylow's first theorem guarantees that there exist subgroups for all prime *powers* that divide 60, so G must have a subgroup of order $2^2 = 4$. Thus by process of elimination, we can conclude that G does not necessarily have a subgroup of order 6.

If you're curious about a concrete example, consider the direct product $A_4 \times \mathbb{Z}_5$. Recall that A_4 is the alternating group on 4 elements, consisting of all the even permutations of 4 elements, of which there are $4!/2 = 12$; this means that $A_4 \times \mathbb{Z}_5$ will contain $12 \cdot 5 = 60$ elements. The elements of A_4 include:

- the identity permutation,
- the 3-cycles (such as (123)), and
- the $2,2$-cycles (such as $(12)(34)$).

These cycle types have orders 1, 3, and 2 respectively, so they once again generate cyclic subgroups of the same order. In addition, taking any two distinct $2,2$-cycles will generate a subgroup of order 4, which is isomorphic to the Klein 4-group V_4. However, if we take any two 3-cycles, or a 3-cycle and a $2,2$-cycle, we will generate the entirety of A_4; therefore the only subgroups of A_4 have order 1, 2, 3, 4, or 12. Notably, A_4 has no subgroup of order 6.

Now, consider $A_4 \times \mathbb{Z}_5$. The identity of $\mathbb{Z}_5$, i.e. 0, has order 1, while the other elements all have order 5. This means that if our subgroup contains any element of the form (a, b) with $a \in A_4$ and $b \in \mathbb{Z}_5 \setminus \{0\}$, then the order of this subgroup will be divisible by 5. Thus if we're going to construct a subgroup of order 6, it must only contain elements of the form $(a, 0)$, which means it must be isomorphic to a subgroup of A_4. However, we've just shown A_4 doesn't contain a subgroup of order 6, so this is impossible.

39. Let $\{a_n\}$ be the sequence defined by $a_n = \left(1 + \frac{(-1)^n}{n}\right)^n$. Calculate $\limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n$.

(A) 0 (B) $\frac{e-1}{e}$ (C) $\frac{e-1}{e^2}$ (D) $\frac{e^2-1}{e}$ (E) $+\infty$

Solution:

Recall that $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. What we have here, then, is an interleaving of two sequences — one is $\left\{\left(1+\frac{1}{n}\right)^n\right\}$, which converges to e, and the other is $\left\{\left(1+\frac{-1}{n}\right)^n\right\}$, which converges to e^{-1}. Therefore $\limsup_{n\to\infty} a_n = e$ and $\liminf_{n\to\infty} a_n = \frac{1}{e}$, giving a difference of $e - \frac{1}{e} = \frac{e^2-1}{e}$.

40. Let $P_3(\mathbb{R})$ be the vector space of all polynomials of degree at most 3 with real coefficients. Consider the following subspaces of $P_3(\mathbb{R})$:

$$U = \{p \in P_3(\mathbb{R}) : p(0) = 0\}$$

$$V = \{p \in P_3(\mathbb{R}) : p(-1) = p(1) = 0\}$$

Which of the following statements are true?

I. $U \cap V$ is a subspace of $P_3(\mathbb{R})$.

II. $U \cup V$ is a subspace of $P_3(\mathbb{R})$.

III. $\dim(U) + \dim(V) = \dim(P_3(\mathbb{R}))$.

(A) I only (B) III only (C) I and II only (D) I and III only (E) I, II, and III

Solution:

Remember that subspaces must be closed under addition and scalar multiplication.

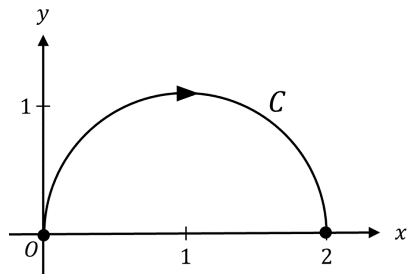
I. An element of $U \cap V$ is a polynomial for which $p(-1) = p(0) = p(1) = 0$. If we add two such polynomials together or multiply one by a real number, the zeros are unchanged, so $U \cap V$ is a subspace.

II. This is not true. Let $p(x) = x$ and $q(x) = x^2 - 1$. Then $p(x) + q(x)$ no longer has any of these roots, so $U \cup V$ is not closed under addition and therefore is not a subspace.

III. Since cubic polynomials have four coefficients, $P_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$, so it has dimension 4. From here, every time we specify one of the roots, we introduce a linear dependency among the coefficients of our polynomials. Suppose an element of $P_3(\mathbb{R})$ is written as $p(x) = ax^3 + bx^2 + cx + d$:

- If $p(0) = 0$, then we know that $d = 0$.
- If $p(1) = 0$, then we know that $a + b + c + d = 0$.
- If $p(-1) = 0$, then we know that $-a + b - c + d = 0$.

Each linear dependency reduces the dimension of our space by 1, so $\dim(U) = 3$ and $\dim(V) = 2$. This means that our given equation is false; the correct statement would be $\dim(U) + \dim(V) - \dim(U \cap V) = \dim(P_3(\mathbb{R}))$, since $\dim(U \cap V) = 1$.



41. Let C be the semicircular path from $(0,0)$ to $(2,0)$ pictured above. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x,y) = (x^2+y^2)\mathbf{i} + 2xy\mathbf{j}$.

(A) 2 (B) $\frac{8}{3}$ (C) 3 (D) 4 (E) $\frac{16}{3}$

Solution:

We could parametrize the semicircle and evaluate our integral directly, but notice what happens when we look at the cross-partials of the components of $\mathbb{F}$:

$$\frac{\partial}{\partial y}(x^2+y^2) = 2y = \frac{\partial}{\partial x}(2xy)$$

This means that we're taking the line integral of a gradient of some function, so the integral result is path-independent by the Fundamental Theorem of Line Integrals. We could instead just choose to integrate over the straight line path from $(0,0)$ to $(2,0)$, or better yet, we could reconstruct the original function (the potential) from its gradient.

$$\int (x^2+y^2)\,dx = \frac{1}{3}x^3 + xy^2 + g(y)$$

$$\frac{\partial}{\partial y}\left(\frac{1}{3}x^3 + xy^2\right) = 2xy + g'(y)$$

Since $g'(y) = 0$, our potential function is $\frac{1}{3}x^3 + xy^2 + C$. Thus the value of our integral is $\left(\frac{1}{3}x^3 + xy^2\right)\Big|_{(0,0)}^{(2,0)} = \frac{8}{3}$.

42. If $y = \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$, then $\dfrac{dy}{dx} =$

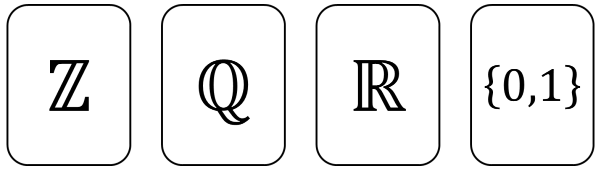
(A) $\dfrac{1}{2y-1}$ (B) $\dfrac{1}{2y+1}$ (C) $\dfrac{y}{2y-1}$ (D) $\dfrac{2y+1}{y}$ (E) $\dfrac{1-2y}{y}$

Solution:

We can rewrite this expression as $y = \sqrt{x+y}$, or $y^2 = x + y$. From here we can use implicit differentiation:

$$\begin{aligned}
\frac{d}{dx}(y^2) &= \frac{d}{dx}(x+y) \\
2y\frac{dy}{dx} &= 1 + \frac{dy}{dx} \\
2y\frac{dy}{dx} - \frac{dy}{dx} &= 1 \\
(2y-1)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{2y-1}
\end{aligned}$$

(*This problem was adapted from a Mu Alpha Theta test*.)



43. Neisha has four index cards, each of which has a different set written on it, as shown above. First, Neisha chooses a card at random, and lets A be the set written on that card. Then, she replaces the card and shuffles the cards, chooses a second card at random, and lets B be the set written on the second card. If F is the set of all functions with domain A and codomain B, what is the probability that F is a countable set?

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{3}{8}$ (D) $\frac{3}{16}$ (E) $\frac{9}{16}$

Solution:

Recall that the number of functions with domain A and codomain B is $|B|^{|A|}$.

The cardinalities of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\{0, 1\}$ are $\aleph_0$, $\aleph_0$, $2^{\aleph_0}$, and 2, respectively. (Also remember that $2^{\aleph}_0$ is strictly greater than $\aleph_0$, which is why we say $\mathbb{R}$ is uncountable.)

The only possible exponentiations that will lead to a cardinality of $\aleph_0$ or less are $2^2 = 4$ and ${\aleph_0}^2 = \aleph_0$; anything else will give an uncountable cardinality. This means that the only combinations that give a countable set of functions are:

$$\{0, 1\} \to \{0, 1\} \qquad \{0, 1\} \to \mathbb{Z} \qquad \{0, 1\} \to \mathbb{Q}$$

Thus 3 of the 16 possible combinations lead to a countable set of functions.

44. Consider the function f defined as

$$f(x) = \begin{cases} \dfrac{c}{x^{5/2}} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1, \end{cases}$$

where c is a real constant. Suppose f is the probability distribution of a continuous random variable X. What is the expected value of X?

(A) 1 (B) 3 (C) 6 (D) 9 (E) ∞

Solution:

In order to be a probability distribution, the area under the pdf curve must be 1. Hence we can integrate to find the value of c:

$$\begin{aligned} \int_1^\infty cx^{-5/2}\,dx = -\frac{2}{3}cx^{-3/2}\Big|_1^\infty &= 1 \\ \frac{2}{3}c &= 1 \\ c &= \frac{3}{2} \end{aligned}$$

Now let's find $\mathrm{E}(X)$:

$$\mathrm{E}(X) = \int_1^\infty xf(x)\,dx = \int_1^\infty \frac{3}{2}x^{-3/2}\,dx = -\frac{3}{\sqrt{x}}\Big|_1^\infty = 3$$

45. Suppose S is a set of continuous functions on $[3, 5]$ such that for each $f \in S$, the following properties hold:

$f(3) = 2$

$f(5) = 4$

$f'(x) > 0$ for all $x \in [3, 5]$

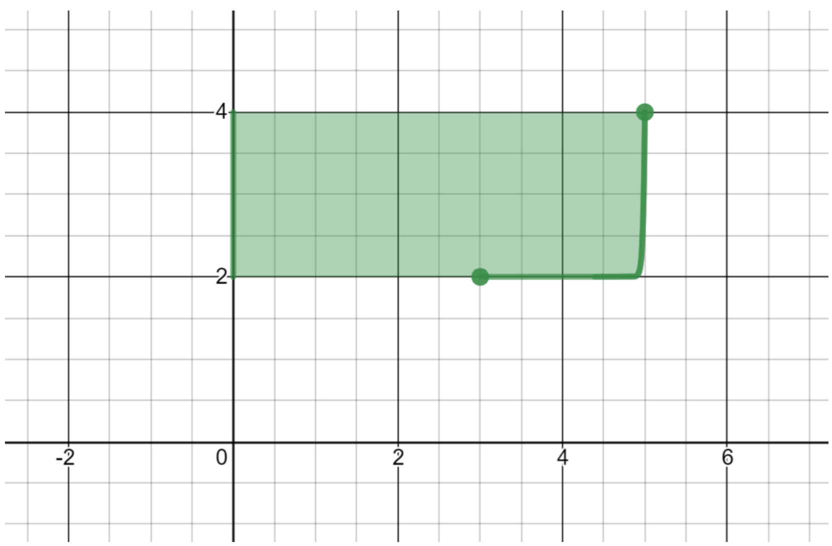
Calculate $\sup_{f \in S} \left\{ \int_2^4 f^{-1}(x)\, dx \right\}$.

(A) 4 (B) 6 (C) 8 (D) 10 (E) 14

Solution:

We can visualize the integral of f^{-1} as measuring area between the curve $y = f(x)$ and the y-axis, in which case we'd really be writing it as $\int_2^4 f^{-1}(y)\, dy$. (Remember, the actual variable name we use doesn't matter — it's just a dummy variable in the end.)

More precisely, if f is an invertible and differentiable function such that $f(a) = c$ and $f(b) = d$, then $\int_a^b f(x)\, dx + \int_c^d f^{-1}(y)\, dy = bd - ac$.



From here, the supremum will be approached as the graph of f looks as much as possible like what's given in the figure above — close to 2 for as long as possible, then shooting up to 4 as x reaches 5. Thus the supremum of all possible areas to the left is 10.

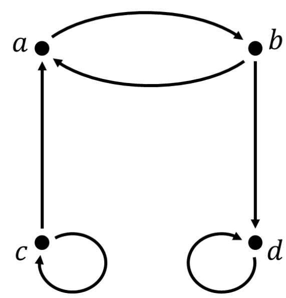
46. For each integer $n \geq 0$, define $P_n(x) = \displaystyle\int_0^x t^n e^{-t}\,dt$. Which of the following recurrences is satisfied by $P_n(x)$ for all $n \geq 1$?

(A) $P_n(x) = nP_{n-1}(x)$
(B) $P_n(x) = x^n e^{-x} + nP_{n-1}(x)$
(C) $P_n(x) = x^n e^{-x} - nP_{n-1}(x)$
(D) $P_n(x) = -x^n e^{-x} + nP_{n-1}(x)$
(E) $P_n(x) = -x^n e^{-x} - nP_{n-1}(x)$

Solution:

Let $u = t^n$ and $dv = e^{-t}\,dt$, in which case $du = nt^{n-1}\,dt$ and $v = -e^{-t}$. Then, using integration by parts, we have:

$$\begin{aligned}
\int_0^x u\,dv &= uv\Big|_0^x - \int_0^x v\,du \\
&= (t^n)(-e^{-t})\Big|_0^x - \int_0^x \left(-e^{-t}\cdot nt^{n-1}\right)\,dt \\
&= -x^n e^{-x} + n\int_0^\infty t^{n-1}e^{-t}\,dt \\
&= -x^n e^{-x} + nP_{n-1}(x)
\end{aligned}$$



47. The vertex-edge graph above depicts a relation $\sim$ on the set $S = \{a, b, c, d\}$. For any $x \in S$ and $y \in S$, an arrow drawn from x to y on the graph signifies that $x \sim y$.

    What is the minimum number of additional arrows that must be drawn so that the relation represented by the resulting vertex-edge graph is transitive?

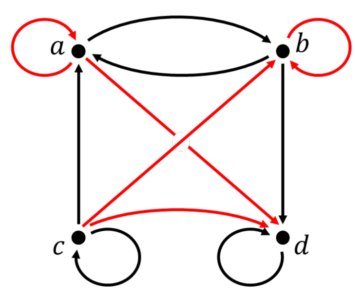
    (A) Two (B) Three (C) Four (D) Five (E) Seven

    Solution:

    We need to make sure that there is an arrow $x \to y$ whenever there is a path from x to y:

    - $a \to b \to a$
    - $b \to a \to b$
    - $c \to a \to b$
    - $a \to b \to d$
    - $c \to a \to b \to d$

    Thus we need five additional arrows, as illustrated below.



48. $\lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{1/n} =$

(A) 1 (B) e (C) $e^{1/e}$ (D) e^e (E) The limit does not exist.

Solution:

We can rewrite our limit as $L = \lim_{n\to\infty} \left(\frac{n}{1} \cdot \frac{n}{2} \cdot \frac{n}{3} \cdot \ldots \cdot \frac{n}{n}\right)^{1/n}$. Taking the logarithm of both sides gives us $\log L = \lim_{n\to\infty} \frac{1}{n} \left(\log \frac{n}{1} + \log \frac{n}{2} + \log \frac{n}{3} + \ldots + \log \frac{n}{n}\right)$. This looks remarkably like a Riemann sum:

$$\begin{aligned}
\log L &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log\left(\frac{n}{k}\right) \\
&= \int_0^1 \log \frac{1}{x}\, dx \\
&= -\int_0^1 \log x\, dx \\
&= -(x \log x - x)\Big|_0^1 = 1
\end{aligned}$$

Since $\log L = 1$, we have $L = e$.

49. Michael brought 12 identical cookies to work. In how many ways can he distribute those cookies to his four coworkers so that each coworker gets at least one cookie?

(A) $\binom{11}{3}$ (B) $\binom{12}{3}$ (C) $\binom{12}{4}$ (D) $\binom{13}{4}$ (E) $\binom{15}{3}$

Solution:

This kind of problem is usually solved via the so-called "stars-and-bars" method, though we have to deal with the condition that each coworker gets at least one cookie. We'll do that by giving one cookie to each coworker ahead of time, which leaves 8 cookies to distribute. Now we think of those cookies as 8 "stars", and we think of our 4 coworkers as being separated by 3 "bars", all of which we can arrange in a line:

$$\star\star\star\star\,|\,\star\,||\,\star\star\star$$

The arrangement above would mean that of the remaining 8 cookies, the four coworkers get 4, 1, 0, and 3, respectively.

The number of arrangements of 8 "stars" and 3 "bars" is $\frac{11!}{8!3!} = \binom{11}{3}$.

50. Estimate $\int_0^1 \frac{\sin t}{t}\,dt$ to the nearest thousandth.

(A) 0.942
(B) 0.943
(C) 0.944
(D) 0.945
(E) 0.946

Solution:

This is what power series are made for! Begin with the Maclaurin series for $\sin t$:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

Divide through by t:

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

Integrate:

$$\int_0^x \frac{\sin t}{t}\,dt = t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!} + \cdots$$

Evaluate at $x = 1$:

$$\begin{aligned}\int_0^1 \frac{\sin t}{t}\,dt &= 1 - \frac{1}{3\cdot 3!} + \frac{1}{5\cdot 5!} - \frac{1}{7\cdot 7!} + \cdots \\ &= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{7\cdot 7!} + \cdots\end{aligned}$$

Since $7! = 5040$, $7\cdot 7!$ will be well above the error bound of 0.0005 we need to be correct to the nearest thousandth, so we can just sum the first three terms. In particular, $\frac{1}{9} = 0.1111\overline{1}\cdots$, so $\frac{1}{18} = 0.0555\overline{5}\cdots$, and $\frac{1}{6} = 0.1666\overline{6}\cdots$, so $\frac{1}{600} = 0.0016\overline{6}\cdots$. Thus we can calculate:

$$\begin{array}{r} 1.0016\overline{6} \\ \underline{-0.0555\overline{5}} \\ 0.9461\overline{1} \end{array}$$

51. Which of the following abelian groups of order 360 is NOT isomorphic to the other four?

(A) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$
(B) $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{15}$
(C) $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$
(D) $\mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_6$
(E) $\mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30}$

Solution:

The direct sum $\mathbb{Z}_m \oplus \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are coprime. This means that, in particular, $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \ncong \mathbb{Z}_4$ (it's actually isomorphic to V_4 instead). Thus choice D is the odd one out. All of the other choices are produced by only combining coprime orders:

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_3 \oplus \mathbb{Z}_5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{15}$$
$$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{10}$$
$$\mathbb{Z}_2 \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3) \oplus (\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{30}$$

52. Let $f : (0,1) \to (0,\infty)$ be a uniformly continuous function. Which of the following statements are true?

I. If $\{x_n\}$ is a Cauchy sequence in $(0,1)$, then $\{f(x_n)\}$ is a Cauchy sequence in $(0,\infty)$.

II. If $\lim\limits_{n\to\infty} x_n$ exists, then $\lim\limits_{n\to\infty} f(x_n) = f\left(\lim\limits_{n\to\infty} x_n\right)$.

III. If $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences in $(0,1)$, then $\{|f(x_n) - f(y_n)|\}$ is a Cauchy sequence in $(0,\infty)$.

(A) I only (B) I and II only (C) I and III only (D) II and III only (E) I, II, and III

Solution:

I. This is a basic fact from real analysis: uniform continuity preserves Cauchy sequences. Let $\varepsilon > 0$; since f is uniformly continuous, there exists a $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Then, since $\{x_n\}$ is Cauchy, there exists an $N > 0$ such that $m, n > N$ implies $|x_m - x_n| < \delta$, which then implies $|f(x_m) - f(x_n)| < \varepsilon$. Thus $\{f(x_n)\}$ is Cauchy.

II. Let $\lim\limits_{n\to\infty} x_n = x$, and let $\varepsilon > 0$. Since f is continuous, there exists a $\delta > 0$ such that $|x_n - x| < \delta$ implies $|f(x_n) - f(x)| < \varepsilon$. By the definition of limit, for this δ, there exists an $N > 0$ such that $n > N$ implies $|x_n - x| < \delta$, which then implies $|f(x_n) - f(x)| < \varepsilon$. Thus $\lim\limits_{n\to\infty} f(x_n) = f(x)$. (This is actually just a property of continuous functions, as long as $\lim\limits_{n\to\infty} f(x_n)$ exists in the first place, which is what uniform continuity guarantees here.)

III. Let $\varepsilon > 0$. By I above, we know that $\{f(x_n)\}$ and $\{f(y_n)\}$ are also Cauchy sequences. This means that we can find an N_1 such that $m, n > N_1$ implies $|f(x_m) - f(x_n)| < \frac{\varepsilon}{2}$ and an N_2 such that $m, n > N_2$ implies $|f(y_m) - f(y_n)| < \frac{\varepsilon}{2}$; letting $N = \max\{N_1, N_2\}$, both implications are satisfied. Now we can use the Triangle Inequality:

$$\begin{aligned}
\big|(f(x_m) - f(y_m)) - (f(x_n) - f(y_n))\big| &= \big|f(x_m) - f(y_m) - f(x_n) + f(y_n)\big| \\
&= \big|(f(x_m) - f(x_n)) - (f(y_m) - f(y_n))\big| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

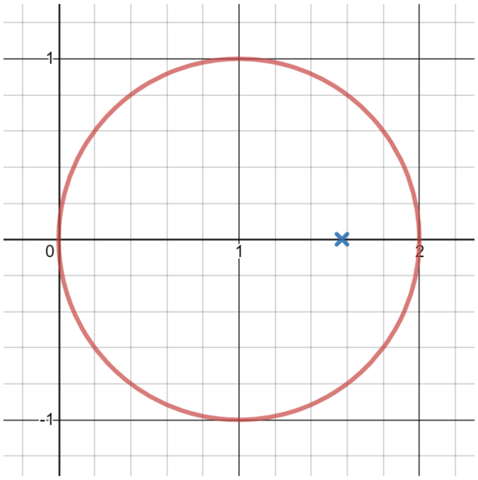
Thus $\{|f(x_n) - f(y_n)|\}$ is a Cauchy sequence as well.

53. Calculate $\dfrac{1}{2\pi i}\displaystyle\oint_C \tan z\, dz$, where C is the circle $|z-1| = 1$ parametrized counterclockwise.

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Solution:

The given circle contains exactly one of the singularities of $\tan z$ at $\dfrac{\pi}{2}$:



This means that we just need to calculate the residue of $\tan z$ at $z_0 = \dfrac{\pi}{2}$. Doing so from the definition would be annoying, but we can use a shortcut formula: If $f(z) = \dfrac{p(z)}{q(z)}$, where $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$, then $\operatorname*{Res}_{z=z_0} f(z) = \dfrac{p(z_0)}{q'(z_0)}$. In this case we have $p(z) = \sin z$ and $q(z) = \cos z$, so $\operatorname*{Res}_{z=\frac{\pi}{2}} \tan z = \dfrac{\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}} = -1$.

From here, we know that $\displaystyle\oint_C \tan z\, dz = 2\pi i \cdot \operatorname*{Res}_{z=\frac{\pi}{2}} \tan z$, so $\dfrac{1}{2\pi i}\displaystyle\oint_C \tan z\, dz = \operatorname*{Res}_{z=\frac{\pi}{2}} \tan z = -1$.

54. Suppose A is a 3×3 matrix with the property that $A^3 = A$. Which of the following must be true?

(A) The eigenvalues of A are distinct.
(B) If A is invertible, then $A^T = A$.
(C) A^2 is the identity matrix or the zero matrix.
(D) The trace of A^2 equals the rank of A.
(E) The absolute value of the trace of A^3 equals the rank of A.

Solution:

Since $A^3 = A$, we have $A^3 - A = A(A+1)(A-1) = 0$; this means that the only eigenvalues of A can be 0, 1, or -1. We can eliminate the incorrect answer choices by carefully constructing example.

- If $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $A^3 = A$, but its only eigenvalue is 0, repeated three times. Eliminate A.
- If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 2 & 0 \end{pmatrix}$, then $A^3 = A$ and $\det A = -1$, but $A^T \neq A$. Eliminate B.
- If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $A^3 = A$, but $A^2 = A$ as well. Eliminate C.

For D and E, there are a few facts that will help us out:

- The trace of a matrix is the sum of its eigenvalues.
- If λ is an eigenvalue of A, then λ^n is an eigenvalue of A_n.
- The rank of a matrix is equal to the sum of its nonzero eigenvalues.

From these, we can see that squaring 0, 1, and -1 will give 0, 1, and 1, respectively; this means that adding up the eigenvalues of A^2 is the same as counting the nonzero eigenvalues of A. This is not so for A^3, which can still have -1 as an eigenvalue. As a result, if $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then the trace of A^3 is 0, while the rank of A is 2.

x	0	1	2	3	4
$f(x)$	1	-5	-7	-2	3

55. Selected values of a polynomial $f(x)$ of degree 4 are given in the table above. What is the value of $f(5)$?

(A) -9 (B) -4 (C) 2 (D) 7 (E) 11

Solution:

Since f is a polynomial of degree 4, its fourth differences $\Delta^4 f(x)$ should be constant. First let's build a table of finite differences:

x	0		1		2		3		4
$f(x)$	1		-5		-7		-2		3
$\Delta f(x)$		-6		-2		5		5	
$\Delta^2 f(x)$			4		7		0		
$\Delta^3 f(x)$				3		-7			
$\Delta^4 f(x)$					-10				

From the table we see that $\Delta^4 f(x) = -10$, so we build the value of $f(5)$ backwards from here:

x	0		1		2		3		4		5
$f(x)$	1		-5		-7		-2		3		-9
$\Delta f(x)$		-6		-2		5		5		-12	
$\Delta^2 f(x)$			4		7		0		-17		
$\Delta^3 f(x)$				3		-7		-17			
$\Delta^4 f(x)$					-10		-10				

Thus $f(5) = -9$.

56. Let $g(z)$ be an analytic function such that $\left|g(z)\right| = 3$ for all z in the open disk $|z| < 2$. If $g(1) = 3i$, find all possible values of $g(-1)$.

(A) $\{3i\}$
(B) $\{-3i\}$
(C) $\{3i, -3i\}$
(D) $\{3, -3, 3i, -3i\}$
(E) $\{z \in \mathbb{C} : |z| = 3\}$

Solution:

In complex analysis, there are plenty of statements that boil down to "if an analytic function has [some property], then it must be constant." One such property is having a constant modulus in an open disk. There are many ways to show this; this is one of them.

Suppose $g(x+iy) = u(x,y)+iv(x,y)$, where u and v are real-valued functions of real variables x and y. Then $|g(x,y)|^2 = u(x,y)^2+v(x,y)^2$, which we are claiming is a constant. This means that their partial derivatives must be zero:

$$\frac{\partial}{\partial x}(u^2+v^2) = 0 \qquad\qquad \frac{\partial}{\partial y}(u^2+v^2) = 0$$

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \qquad\qquad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \qquad\qquad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

Now, the Cauchy-Riemann equations tell us that for an analytic function, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. We can use these to rewrite our equations above entirely in terms of v:

$$u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial x} = 0 \qquad\qquad -u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

Multiplying the first equation by u and the second equation by v and then adding them together, we end up with:

$$(u^2+v^2)\frac{\partial v}{\partial y} = 0$$

If $u^2+v^2 = 0$, then $g(z) = 0$, so g is constant. Otherwise, if $\frac{\partial v}{\partial y} = 0$, which means that v is constant and therefore $\frac{\partial v}{\partial x} = 0$ as well. Applying the Cauchy-Riemann equations once more, we can show that u is constant as well, and that therefore g must be constant.

Thus we must have $g(-1) = 3i$.

57. Let C_1 be the curve defined by the polar equation $r = \dfrac{\theta}{\theta+1}$ for $\theta \geq 0$, and let C_2 be the circle defined by the equation $x^2 + y^2 = 1$. Let $C = C_1 \cup C_2$.

Which of the following statements are true with respect to the standard topology on $\mathbb{R}^2$?

I. For any point $P \in C$, there exists an open disk centered at P containing some point $Q \in C$ such that $P \neq Q$.

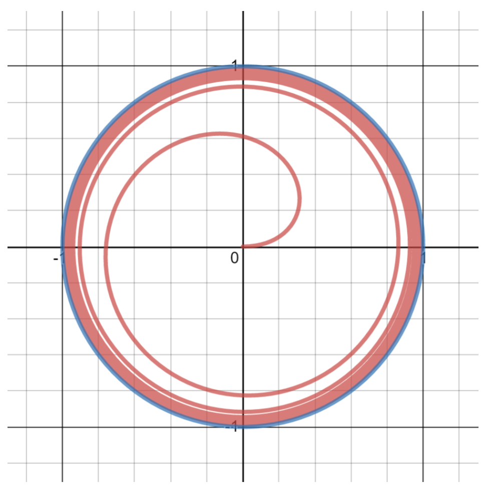
II. For any collection $\mathcal{F}$ of open disks such that $C \subseteq \bigcup \mathcal{F}$, there exists a finite subcollection $\mathcal{G} \subseteq \mathcal{F}$ such that $C \subseteq \bigcup \mathcal{G}$.

III. For any pair of points $P \in C$ and $Q \in C$, there exists a continuous function $f : [0,1] \to C$ such that $f(0) = P$ and $f(1) = Q$.

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

Solution:

The graph of C_1 is a spiral that approaches the unit circle C_2, as shown in the figure below. The resulting shape is sometimes called the "topologist's whirlpool", and it's very similar to the well-known "topologist's sine curve."



I. This is asking whether every point of C is a limit point of C; this is true because C has no isolated points. (We would call C a perfect set.)

II. This is asking whether C is compact. Since C is the closure of C_1, it is therefore closed; since C is also bounded, the Heine-Borel theorem tells us that C is compact with respect to the standard topology on $\mathbb{R}^2$.

III. This is asking whether C is path-connected; however, much like the topologists's sine curve, while C is connected, it is NOT path-connected. Actually proving this rigorously on the exam isn't necessary — you just need to see the similarities between the curves and make a quick judgment. If you want to see a proof, though, there's one provided in *Introduction to Topology, Applied and Pure* by Adams and Franzosa.

58. $\displaystyle\int_0^\infty \frac{1}{1+e^{ax}}\,dx =$

(A) $\dfrac{1}{a}$ (B) $a\log 2$ (C) $\dfrac{1}{a\log 2}$ (D) $\dfrac{\log 2}{a}$ (E) $\dfrac{a}{\log 2}$

Solution:

Letting $u = e^{ax}$, we have $du = ae^{ax}\,dx$. We don't have an extra e^{ax} anywhere, but that's fine; we can make the same substitution again and write $du = au\,dx$, or $\dfrac{du}{au} = dx$. Hence we have:

$$\int_0^\infty \frac{1}{1+e^{ax}}\,dx = \int_1^\infty \frac{1}{1+u}\cdot\frac{du}{au} = \frac{1}{a}\int_1^\infty \frac{1}{u(u+1)}\,du$$

The fraction $\dfrac{1}{u(u+1)}$ can be shown via partial fractions to equal $\dfrac{1}{u} - \dfrac{1}{u+1}$, which we can use to integrate:

$$\begin{aligned}\frac{1}{a}\int_1^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du &= \frac{1}{a}\big(\log u - \log(u+1)\big)\Big|_1^\infty \\ &= \frac{1}{a}\log\frac{u}{u+1}\Big|_1^\infty \\ &= \frac{1}{a}\left(\log 1 - \log\frac{1}{2}\right) \\ &= \frac{1}{a}(0+\log 2) \\ &= \frac{\log 2}{a}\end{aligned}$$

A slicker way to do this would be to multiply the top and bottom through by e^{-ax}.

59. To the nearest thousand, approximately how many roots does the function $f(x) = e^{-x}\sin(x^2)$ have on the interval $[0, 100]$?

(A) 1000 (B) 2000 (C) 3000 (D) 4000 (E) 5000

Solution:

The zeros of $\sin(x^2)$ will occur at the square roots of the zeros of $\sin x$: $x = 0$, $\sqrt{\pi}$, $\sqrt{2\pi}$, $\sqrt{3\pi}$, and so on. Therefore we need to estimate the value of n such that $\sqrt{n\pi} \approx 100$.

$$\begin{aligned}
\sqrt{n\pi} &\approx 100 \\
n\pi &\approx 10\,000 \\
n &\approx \frac{10\,000}{\pi} \\
&\lessapprox \frac{10\,000}{3} \\
&\approx 3\,333
\end{aligned}$$

60. Circle C is tangent to the graph of $y = x^2$ at the origin and has the same curvature as the parabola at the point of tangency. What is the radius of circle C?

(A) $\frac{1}{3}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{3}{2}$ (E) 2

Solution:

Every GRE has a problem or two that uses some random formula you've likely forgotten. In this case, it's the curvature formula.

One common formula for curvature (that doesn't require that we reparametrize in terms of arclength) is:

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Parametrizing our parabola as $\mathbf{r}(t) = (t, t^2)$, we have $\mathbf{r}'(t) = (1, 2t)$, so $\|\mathbf{r}'(t)\| = \sqrt{1+4t^2}$. Then we can calculate $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \left(\frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}\right)$; differentiating again, we end up with $\mathbf{T}'(t) = \left(-\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}\right)$.

Now $\mathbf{r}'(0) = (1, 0)$ and $\mathbf{T}'(0) = (0, 2)$, so $\kappa = \frac{\|\mathbf{T}'(0)\|}{\|\mathbf{r}'(0)\|} = 2$. The osculating circle therefore has radius $r = \frac{1}{\kappa} = \frac{1}{2}$.

However, since we're working in 2D, there's an even faster formula we can use when y is a function of x:

$$\kappa = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

Since $y' = 2x$ and $y'' = 2$, this becomes:

$$\kappa = \frac{|2|}{(1+(0)^2)^{3/2}} = 2$$

Again we have $\kappa = 2$, so $r = \frac{1}{2}$.

61. Let V be the vector space of continuous functions $\mathbb{C} \to \mathbb{C}$ under pointwise addition and scalar multiplication by complex numbers, and let A be the set of fourth roots of unity in the complex plane. For each $\alpha \in A$, define the set $S_\alpha = \{\cos \alpha z, \sin \alpha z, e^{\alpha z}\}$.

What is the dimension of the subspace of V spanned by $\bigcup_{\alpha \in A} S_\alpha$?

(A) 3 (B) 4 (C) 6 (D) 8 (E) 12

Solution:

The fourth roots of unity are $A = \{1, i, -1, -i\}$, so the sets S_α are:

$$\begin{aligned} S_1 &= \{\cos z, \sin z, e^z\} \\ S_i &= \left\{\cos iz, \sin iz, e^{iz}\right\} = \left\{\cosh z, i \sinh z, e^{iz}\right\} \\ S_{-1} &= \left\{\cos(-z), \sin(-z), e^{-z}\right\} = \left\{\cos z, -\sin z, e^{-z}\right\} \\ S_{-i} &= \left\{\cos(-iz), \sin(-iz), e^{-iz}\right\} = \left\{\cosh z, -i \sinh z, e^{-iz}\right\} \end{aligned}$$

From here, we notice immediately that by ignoring complex scalar multiples, our subspace is spanned by eight functions:

$$\left\{\sin z, \cos z, \sinh z, \cosh z, e^z, e^{-z}, e^{iz}, e^{-iz}\right\}$$

However, we can remove even more redundancy using Euler's identity:

$$\begin{aligned} e^{iz} &= \cos z + i \sin z \\ e^{-iz} &= \cos(-z) + i \sin(-z) = \cos z - i \sin z \\ e^z &= \cos(-iz) + i \sin(-iz) = \cosh z + \sinh z \\ e^{-z} &= \cos(iz) + i \sin(iz) = \cosh z - \sinh z \end{aligned}$$

Thus we only need four elements to span this space.

(If we REALLY wanted to be sure we can't span the space with even fewer, we could calculate the Wronskian, but that would be overkill at this point. That being said, if you're bored one day, you're welcome to work that out ... you should get 4.)

62. Which of the following must be true of a function $f : \mathbb{R}^2 \to \mathbb{R}$?

I. If $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist and are continuous at $(0,0)$, then f is differentiable there.

II. If f has directional derivatives in all directions at $(0,0)$, then f is differentiable there.

III. If $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ exist at $(0,0)$, then they are equal there.

(A) None (B) II only (C) I and II only (D) I and III only (E) II and III only

Solution:

I. Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Then f is identically zero along both the x-axis and the y-axis, so $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$. However, the limit of f along the line $y = x$ is $\dfrac{1}{2}$, so f is not even continuous, much less differentiable.

II. Let $f(x,y) = \sqrt[3]{x^2 y}$. Then if $\mathbf{u} = (\cos\theta, \sin\theta)$ is a unit vector, the directional derivative of f at $(0,0)$ is $f'_{\mathbf{u}}(0,0) = \lim_{h\to 0} \dfrac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} = \sqrt[3]{\cos^2\theta \sin\theta}$, which exists for any θ. We can get the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ in particular by letting $\theta = 0$ and $\theta = \dfrac{\pi}{2}$, respectively, both of which give 0. However, this would imply that the directional derivative equals $\nabla f(0,0) \cdot \mathbf{u} = 0$, but our expression in terms of θ is not always 0. Thus f is not differentiable.

To visualize f, see `https://www.math.tamu.edu/~tom.vogel/gallery/node17.html`.

III. Let $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$. Then, using the quotient rule, the partial derivatives are:

$$\frac{\partial f}{\partial x} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \qquad \frac{\partial f}{\partial y} = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

Notice that setting $x = 0$ gives us $f_x(0,y) = -y$, so $f_x(0,0) = \lim_{x\to 0} \dfrac{f(x,0) - f(0,0)}{x} = 0$. Similarly, $f_y(x,0) = x$ and $f_y(0,0) = 0$. Differentiating with respect to the other variable, we have $\dfrac{\partial^2 f}{\partial y \partial x}\Big|_{(x,y)=(0,0)} = f'_{xy}(0,0) = \lim_{y\to 0} \dfrac{f_x(0,y) - f_x(0,0)}{y} = -1$, but $\dfrac{\partial^2 f}{\partial x \partial y}\Big|_{(x,y)=(0,0)} = f'_{yx}(0,0) = \lim_{x\to 0} \dfrac{f_x(x,0) - f_x(0,0)}{x} = 1$. Thus the mixed partials are not equal despite both being defined.

63. Consider the matrix equation $QR\mathbf{x} = \mathbf{b}$, where $Q = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ is an orthogonal matrix, $R = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{pmatrix}$ is an upper triangular matrix, and $\mathbf{b} = \begin{pmatrix} 11 \\ -10 \\ -65 \end{pmatrix}$.

If $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, what is the value of x_3?

(A) -6 (B) -3 (C) 3 (D) 9 (E) 12

Solution:

Since Q is orthogonal, we have $Q^{-1} = Q^T$, so multiplying this on both sides of the equation we have $R\mathbf{x} = Q^T\mathbf{b}$:

$$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 11 \\ -10 \\ -65 \end{pmatrix}$$

We can get the value of x_3 by multiplying the last row of each matrix by the column vector on each side of the equation:

$$\begin{aligned} 0x_1 + 0x_2 + 9x_3 &= -\tfrac{2}{3}(11) + \tfrac{1}{3}(-10) + \tfrac{2}{3}(-65) \\ 9x_3 &= -54 \\ x_3 &= -6 \end{aligned}$$

64. Let A and B be ideals of a ring R, and define the ideals $A+B$ and AB as follows:

$$A + B = \{a + b : a \in A, b \in B\}$$
$$AB = \{a_1 b_1 + \cdots + a_n b_n : a_i \in A, b_i \in B, i \in \{1, \ldots, n\}, n \in \{1, 2, \ldots\}\}$$

Which of the following correctly orders $A+B$, AB, and $A \cap B$ via inclusion?

(A) $AB \subseteq A+B \subseteq A \cap B$
(B) $A \cap B \subseteq AB \subseteq A+B$
(C) $A \cap B \subseteq A+B \subseteq AB$
(D) $AB \subseteq A \cap B \subseteq A+B$
(E) $A+B \subseteq A \cap B \subseteq AB$

Solution:

Rather than thinking about this in the abstract, the easiest way to handle this would be to look at the ideals of a very familiar ring — $\mathbb{Z}$. Let $A = 4\mathbb{Z}$ and $B = 6\mathbb{Z}$. Then:

- $A+B$ is the set of all integers that can be written as $4x+6y$; since $\gcd(4,6) = 2$, we have $A+B = 2\mathbb{Z}$.
- AB is the set of all integers that can be written as $4x_1 \cdot 6y_1 + \ldots + 4x_n \cdot 6y_n$. We can simplify this to $24(x_1 y_1 + \ldots + x_n y_n)$; therefore $AB = 24\mathbb{Z}$.
- $A \cap B$ is the set of all integers that are divisible by both 4 and 6; since $\operatorname{lcm}(4,6) = 12$, we have $A \cap B = 12\mathbb{Z}$.

Thus the correct order of inclusion is $AB \subseteq A \cap B \subseteq A+B$. (Remember, when n is smaller, $n\mathbb{Z}$ is a "larger" ideal because it contains more elements.)

65. For each pair of integers a and b, let the set $\{a + bn : n \in \mathbb{Z}\}$ be denoted $a + b\mathbb{Z}$. Consider the topology $\mathcal{T}$ on $\mathbb{Z}$ whose basis is $\{a + b\mathbb{Z} : a, b \in \mathbb{Z} \text{ and } b \neq 0\}$. Which of the following statements is false?

(A) $(\mathbb{Z}, \mathcal{T})$ is Hausdorff.
(B) $(\mathbb{Z}, \mathcal{T})$ is totally disconnected.
(C) The set $\{0\}$ is closed under $\mathcal{T}$.
(D) No nonempty open sets of $\mathcal{T}$ are finite.
(E) Exactly two sets of $\mathcal{T}$ are both open and closed.

Solution:

If you'd like to look this topology up later, it's called the "evenly spaced integer topology" or "arithmetic progression topology."

A: $(\mathbb{Z}, \mathcal{T})$ is Hausdorff if for any two elements $x, y \in \mathbb{Z}$, we can find two disjoint open sets that "separate" them. In this case, all we need to do is find an integer n that does not divide $y - x$, and then the sets $x + n\mathbb{Z}$ and $y + n\mathbb{Z}$ will separate them.

B: $(\mathbb{Z}, \mathcal{T})$ is totally disconnected if any set with at least two elements is disconnected. Suppose we have two elements $x, y \in S \subseteq \mathbb{Z}$, and again find an integer n that does not divide $y - x$. Then let $U = x + n\mathbb{Z}$ and $V = (x + 1 + n\mathbb{Z}, \cdots, x + (n-1) + n\mathbb{Z})$ — in other words, V is the union of all arithmetic sequences with difference n *other* than U. Then $U \cap V = \varnothing$, while $(U \cap S) \cup (V \cap S) = S$, so S must be disconnected.

C: $\{0\}$ is closed if its complement $\mathbb{Z} \setminus \{0\}$ is open. Consider the following open sets:

$$1 + 2\mathbb{Z}, 2 + 4\mathbb{Z}, 4 + 8\mathbb{Z}, 8 + 16\mathbb{Z}, \cdots$$

The union of these open sets is $\mathbb{Z} \setminus \{0\}$. Since the union of any number of open sets in a topology is open, $\mathbb{Z} \setminus \{0\}$ is open, so $\{0\}$ must be closed.

D: Since the basis elements are all infinite, the only way we could possibly get a finite open set is by an intersection. It suffices to consider what happens when intersecting basis elements: if two basis elements $a + b\mathbb{Z}$ and $c + d\mathbb{Z}$ have at least one element k in common, then they will again overlap after any multiple of $\operatorname{lcm}(b, d)$, so we can explicitly calculate that $(a + b\mathbb{Z}) \cap (c + d\mathbb{Z}) = k + \operatorname{lcm}(b, d)\mathbb{Z}$. Since we're only allowed to take finite intersections, all open sets must be infinite.

E: This may be true on the standard topology on $\mathbb{R}$, but it's not true here! For example, the set $1 + 2\mathbb{Z}$ is open since it's a basis element, but since its complement $0 + 2\mathbb{Z}$ is also open, $1 + 2\mathbb{Z}$ is closed as well. In fact, it can be shown that *any* basis element is both open and closed!

66. Let an abelian group M form a left module over a ring R. We say that a subset S of M is a spanning set of M if for every $m \in M$, there exist $\{r_1, r_2, \ldots, r_n\} \subseteq R$ and $\{s_1, s_2, \ldots, s_n\} \subseteq S$ such that

$$m = \sum_{i=1}^{n} r_i s_i.$$

We call this spanning set S minimal if no proper subset of S is a spanning set for S. In addition, S is called a basis for M if $m = 0$ implies $r_1 = r_2 = \cdots = r_n = 0$.

Which of the following statements must be true?

I. If R is finite, then any basis S of M is finite.

II. If S and S' are two distinct minimal spanning sets of M, then $|S| = |S'|$.

III. If R is a field and S is a finite minimal spanning set of M, then S is a basis of M.

(A) II only (B) III only (C) I and II only (D) I and III only (E) II and III only

Solution:

You can think of a module intuitively as being just like a vector space, with the elements of the group M being like the "vectors," except instead of the "scalar" coefficients coming from a *field*, they come from a *ring*.

I. One example of a module is $R[x]$, the set of all polynomials with coefficients in a ring R. Even if R is finite — say, perhaps $R = \mathbb{Z}_5$ — the basis $\{1, x, x^2, x^3, \cdots\}$ is infinite.

II. Another example of a module is any abelian group, which can be turned into a module using $\mathbb{Z}$ as the ring of coefficients. In fact, we can even make the group be $\mathbb{Z}$ as well, so we're just multiplying and adding integers.

Now, an easy spanning set for $\mathbb{Z}$ would be $S = \{1\}$, since every integer can be written as $a \cdot 1$ for some $a \in \mathbb{Z}$. This spanning set is minimal, since if you take 1 out, there's nothing left!

A different spanning set would be $S' = \{2, 3\}$, since $\gcd(2, 3) = 1$ and therefore every integer can be written as $a \cdot 2 + b \cdot 3$ for some $a, b \in \mathbb{Z}$. This spanning set is also minimal, since removing 2 would only span $3\mathbb{Z}$ and removing 3 would only span $2\mathbb{Z}$.

However, $|S| \neq |S'|$. This is a way in which modules are different from vector spaces — they don't necessarily have the *invariant basis number condition*.

III. If R is a field, then M is actually a vector space after all! Vector spaces *do* have the invariant basis number condition. Since M has a finite minimal spanning set S, that set S must be a basis. (It's a good thing we have the finiteness condition given, because otherwise we'd need to invoke Zorn's Lemma to claim that every vector space has a basis!)

Many thanks to the folks at `https://discord.sg/math` *for helping me put together this question!*